cone 領域における反応拡散方程式系の時間大域解の非存在について

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We consider nonnegative solutions of initial-boundary value problems for the reaction-diffusion systems of the form

$$\begin{cases} u_t = \Delta u + K_1(x,t)v^{p_1}, & x \in D,\ t > 0, \\ v_t = \Delta v + K_2(x,t)u^{p_2}, & x \in D,\ t > 0, \\ u(x,t) = v(x,t) = 0, & x \in \partial D,\ t > 0, \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x), & x \in D, \end{cases} \tag{1}$$

where $p_1, p_2 \geq 1$ with $p_1 p_2 > 1$. The domain D is a cone in $\mathbf{R}^N$, such as

$$D = \{x \in \mathbf{R}^N; x \neq 0 \text{ and } x/|x| \in \Omega\}, \tag{2}$$

where Ω is some region on S^{N-1} smooth enough.

The initial data $u_0(x)$ and $v_0(x)$ are nonnegative, bounded and continuous in $\bar{D}$, and $u_0(x) = v_0(x) = 0$ on ∂D. The inhomogeneous terms K_i $(i = 1, 2)$ are nonnegative continuous functions in $D \times (0, \infty)$.

In this paper we denote by BC the set of all bounded continuous functions in $\bar{D}$. The "nontrivial solution" denotes the solution u satisfying $(u, v) \not\equiv 0$ in $D \times (0, T)$ with some $T > 0$, it thus means that $(u_0, v_0) \not\equiv 0$ with the condition $(u_0, v_0) \in BC$.

For the Laplace-Beltrami operator with homogeneous Dirichlet boundary condition on $\Omega \in S^{N-1}$, define ω_n as Dirichlet eigenvalues and $\psi_n(\theta)$ as the Dirichlet eigenfunctions corresponding to ω_n which is normalized so that

$$\int_\Omega \psi_n(\theta) d\theta = 1.$$

It is following that

$$\int_\Omega \psi_m(\theta)\psi_n(\theta) d\theta = 0$$

for $m \neq n$. We introduce the Green's function $G(x, y, t) = G(r, \theta, \rho, \phi, t)$ for the linear heat equation in the cone D, where

$$r = |x|,\ \rho = |y|,\ \theta = x/|x| \text{ and } \phi = y/|y| \in \Omega \tag{3}$$

The Green's function is expressed to

$$G(r, \theta, \rho, \phi, t) = \frac{1}{2t}(r\rho)^{-(N-2)/2} \exp\left(-\frac{\rho^2 + r^2}{4t}\right) \sum_{n=1}^{\infty} I_{\nu_n}\left(\frac{r\rho}{2t}\right) \psi_n(\theta)\psi_n(\phi), \tag{4}$$

where $\nu_n = [(N-2)^2/4+\omega_n]^{1/2}$, and I_ν is the modified Bessel function or

$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(z/2)^{2k}}{k!\Gamma(\nu+k+1)} \tag{5}$$

with the Gamma function $\Gamma(z) = \int_0^\infty s^{z-1}e^{-s}ds$ (see Watson [27, p.p.395]).

For our first theorem we shall give the conditions of the inhomogeneous terms K_i $(i = 1, 2)$ as following:

$$\left.\begin{array}{c} \text{there exist } C_U, \hat{\sigma}_i \text{ and } \hat{q}_i \geq 0 \text{ such that} \\ K_i(x,t) \leq C_U\langle x\rangle^{\hat{\sigma}_i}(t+1)^{\hat{q}_i} \quad \text{for any } x\in D,\ t\geq 0, \end{array}\right\} \tag{6}$$

where $\langle x\rangle = (|x|^2+1)^{1/2}$.

Let L^∞_a be a Banach space of L^∞-functions in D with the norm

$$\|\xi\|_{\infty,a} \equiv \text{esssup}_{x\in D}(\langle x\rangle^a|\xi(x)|).$$

For $T > 0$, set

$$E_T = \{(u,v) : [0,T] \to L^\infty_{\delta_1} \times L^\infty_{\delta_2}; \|(u,v)\|_{E_T} < \infty\} \tag{7}$$

with the norm

$$\|(u,v)\|_{E_T} := \sup_{t\in[0,T]} \{\|u(t)\|_{\infty,\delta_1} + \|v(t)\|_{\infty,\delta_2}\},$$

where

$$\delta_i = \frac{\hat{\sigma}_j p_i + \hat{\sigma}_i}{p_i p_j - 1} \quad ((i,j) = (1,2), (2,1)). \tag{8}$$

It is easily seen that E_T is a Banach space.

We begin with stating the existence of the local solution for (1).

Theorem 1. *Assume that u_0, $v_0 \in BC$, $u_0 \equiv v_0 \equiv 0$ on ∂D, and $\langle x\rangle^{\delta_1}u_0(x)$, $\langle x\rangle^{\delta_2}v_0(x)$ are bounded in $\bar{D}$. Suppose that $K_i(x,t)$ $(i = 1,2)$ satisfy* (6)*. Then there exists a nonnegative solution $(u,v) \in E_T$ which solves* (1) *in $D \times (0,T)$ for some $T > 0$.*

For given initial values (u_0, v_0), let $T^* = T^*(u_0, v_0)$ be a maximal existence time of the solution of (1). If $T^* = \infty$, the solutions are global in time. On the other hand, if $T^* < \infty$, then the solutions are not global in time. If the solution blows up in finite time such that

$$\limsup_{t\to T^*} \|u(\cdot,t)\|_\infty + \limsup_{t\to T^*} \|v(\cdot,t)\|_\infty = \infty, \tag{9}$$

then the solution is not global, where $\|\cdot\|_\infty$ denotes the L^∞-norm with respect to space variable.

For our second theorem we shall define a region $\tilde{D}$ such that

$$\left.\begin{array}{c}\text{there exist } k > 0 \text{ and } \{x_m\}_{m=1}^{\infty} \text{ satisfying } 0 < |x_m| < |x_{m+1}|, \\ B(x_m, k|x_m|) \subset \tilde{D} \subset D \text{ for any } m, \text{ and } \lim_{m\to\infty} |x_m| = \infty,\end{array}\right\} \tag{10}$$

where $B(x,r)$ denotes the ball with radius r centered at x. We let the inhomogeneous terms K_i $(i = 1, 2)$ satisfy

$$\left.\begin{array}{c}\text{there exist } C_L > 0, \sigma_i, q_i \geq 0 \text{ and } \tilde{D} \text{ satisfying (10) such that} \\ K_i(x,t) \geq C_L|x|^{\sigma_i} t^{q_i} \quad \text{ for any } x \in \tilde{D},\ t \geq 0.\end{array}\right\} \tag{11}$$

For the theorem we should define γ_+ denoting the positive root of the equation $\gamma(\gamma+N-2) = \omega_1$,

$$\alpha_i = \frac{(2+\sigma_i+2q_i)+(2+\sigma_j+2q_j)p_i}{p_i p_j - 1} \quad ((i,j) = (1,2),(2,1)), \tag{12}$$

and

$$H_a = \{\xi \in C(\bar{D}); \xi(x) \geq M\langle x\rangle^{-a}\psi_1(x/|x|) \text{ for } x \in \tilde{D} \text{ with some } M > 0\}.$$

The main result of this paper is summarized in the following theorem.

Theorem 2. *Assume that* $u_0,\ v_0 \in BC$*,* $u_0 \equiv v_0 \equiv 0$ *on* ∂D*, and* $K_i(x,t)$ $(i = 1, 2)$ *satisfy* (11)*. Suppose that one of the following two conditions holds;*

(i) $\max\{\alpha_1, \alpha_2\} \geq N + \gamma_+$.

(ii) $u_0 \in H_{a_1}$ *with* $a_1 < \alpha_1$ *or* $v_0 \in H_{a_2}$ *with* $a_2 < \alpha_2$.

Then, there exists no nontrivial nonnegative global solution of (1).

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