Analytic semigroups generated by generalized
Ornstein-Uhlenbeck operators

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We consider the operator

\[ A_{\Phi,G,V}u := \Delta u - \nabla \Phi \cdot \nabla u + G \cdot \nabla u - Vu \]

on the weighted space \( L^p(\mathbb{R}^N, \mu) \), \( 1 < p < \infty \), where \( \mu(dx) = e^{-\Phi}dx \). Under the following assumption on \( \Phi : \mathbb{R}^N \to [0, \infty) \), \( G : \mathbb{R}^N \to \mathbb{R}^N \) and \( V : \mathbb{R}^N \to [0, \infty) \) we show that \( A_{\Phi,G,V} \) with domain \( W^{2,p}(\mathbb{R}^N, \mu) \) (for the definition see below) generates an analytic semigroup on \( L^p(\mathbb{R}^N, \mu) \).

(A1) \( \Phi \in C^2(\mathbb{R}^N, \mathbb{R}) \), \( G \in C^1(\mathbb{R}^N, \mathbb{R}^N), \) \( V \in C^1(\mathbb{R}^N, \mathbb{R}) \) and \( \int_{\mathbb{R}^N} e^{-\Phi(x)} \, dx < \infty \).

(A2) For each \( \varepsilon > 0 \) there exists a constant \( C_\varepsilon > 0 \) such that

\[ |\div G| + |D^2\Phi| \leq \varepsilon(|\nabla \Phi|^2 + V) + C_\varepsilon. \]

(A3) There exists a constant \( \beta \in \mathbb{R} \) such that \( G \cdot \nabla \Phi - \div G - V \leq \beta \).

(A3)' \( \div G = G \cdot \nabla \Phi - V \).

(A4) There exists a constant \( \gamma > 0 \) such that \( |G| \leq \gamma(|\nabla \Phi| + V^{1/2} + 1) \).

(A5) For each \( \lambda > 0 \) there exists a constant \( K_\lambda > 0 \) such that \( |\nabla V| \leq \lambda V^{3/2} + K_\lambda \).

**Definition.** Define the space \( W^{k,p}_{V}(\mathbb{R}^N, \mu) \) as

\[ W^{k,p}_{V}(\mathbb{R}^N, \mu) := \left\{ u \in W^{k,p}_{\text{loc}}(\mathbb{R}^N) ; D^\alpha u \in L^p(\mathbb{R}^N, \mu) \text{ if } |\alpha| \leq k, \, Vu \in L^p(\mathbb{R}^N, \mu) \right\}, \]

\[ \|u\|_{W^{k,p}_{V}(\mathbb{R}^N, \mu)} := \|u\|_{W^{k,p}(\mathbb{R}^N, \mu)} + \|Vu\|_{L^p(\mathbb{R}^N, \mu)}. \]

Note that \( C^0(\mathbb{R}^N) \) is dense in \( W^{k,p}_{V}(\mathbb{R}^N, \mu) \).

**Theorem.** Assume that conditions (A1), (A2), (A3), (A4) and (A5) are satisfied. Then the operator

\[ A_{\Phi,G,V} = \Delta - \nabla \Phi \cdot \nabla + G \cdot \nabla - V \]

with domain \( D(A_{\Phi,G,V}) = W^{2,p}_{V}(\mathbb{R}^N, \mu) \) generates an analytic semigroup \( T(t) \) on \( L^p(\mathbb{R}^N, \mu) \), \( 1 < p < \infty \), such that \( \|T(t)\|_{L^p(\mathbb{R}^N, \mu)} \leq e^{t\beta/p} \), where \( \mu(dx) = e^{-\Phi(x)}dx. \) Furthermore, \( \mu \) is an invariant measure of \( T(t) \), that is,

\[ \int_{\mathbb{R}^N} T(t)f \mu(dx) = \int_{\mathbb{R}^N} f \mu(dx) \quad \text{for } f \in C_b(\mathbb{R}^N), \, t \geq 0, \]

if and only if (A3)' holds in addition, where \( C_b(\mathbb{R}^N) \) is the space of bounded continuous functions. Moreover, \( T(t) \) is symmetric if \( p = 2 \) and \( G = 0 \).

This theorem generalizes the result for the case where \( V \equiv 0 \) obtained by [1].

**References**