Analytic semigroups generated by generalized Ornstein-Uhlenbeck operators

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We consider the operator

$$A_{\Phi,G,V}u := \Delta u - \nabla\Phi\cdot\nabla u + G\cdot\nabla u - Vu$$

on the weighted space $L^p(\mathbb{R}^N,\mu)$, $1<p<\infty$, where $\mu(dx)=e^{-\Phi}dx$. Under the following assumption on $\Phi:\mathbb{R}^N\to[0,\infty)$, $G:\mathbb{R}^N\to\mathbb{R}^N$ and $V:\mathbb{R}^N\to[0,\infty)$ we show that $A_{\Phi,G,V}$ with domain $W^{2,p}_V(\mathbb{R}^N,\mu)$ (for the definition see below) generates an analytic semigroup on $L^p(\mathbb{R}^N,\mu)$.

(A1) $\Phi\in C^2(\mathbb{R}^N,\mathbb{R}), G\in C^1(\mathbb{R}^N,\mathbb{R}^N), V\in C^1(\mathbb{R}^N,\mathbb{R})$ and $\int_{\mathbb{R}^N} e^{-\Phi(x)}\,dx<\infty$.
(A2) For each $\varepsilon>0$ there exists a constant $C_\varepsilon>0$ such that

$$|\operatorname{div} G| + |D^2\Phi| \le \varepsilon(|\nabla\Phi|^2+V)+C_\varepsilon.$$

(A3) There exists a constant $\beta\in\mathbb{R}$ such that $G\cdot\nabla\Phi - \operatorname{div} G - V\le\beta$.
(A3)′ $\operatorname{div} G = G\cdot\nabla\Phi - V$.
(A4) There exists a constant $\gamma>0$ such that $|G|\le\gamma(|\nabla\Phi|+V^{1/2}+1)$.
(A5) For each $\lambda>0$ there exists a constant $K_\lambda>0$ such that $|\nabla V|\le\lambda V^{3/2}+K_\lambda$.

Definition. Define the space $W^{k,p}_V(\mathbb{R}^N,\mu)$ as

$$W^{k,p}_V(\mathbb{R}^N,\mu) := \left\{u\in W^{k,p}_{\text{loc}}(\mathbb{R}^N)\ ;\ D^\alpha u\in L^p(\mathbb{R}^N,\mu) \text{ if } |\alpha|\le k,\ Vu\in L^p(\mathbb{R}^N,\mu)\right\},$$
$$\|u\|_{W^{k,p}_V(\mathbb{R}^N,\mu)} := \|u\|_{W^{k,p}(\mathbb{R}^N,\mu)} + \|Vu\|_{L^p(\mathbb{R}^N,\mu)}.$$

Note that $C^\infty_0(\mathbb{R}^N)$ is dense in $W^{k,p}_V(\mathbb{R}^N,\mu)$.

Theorem. *Assume that conditions* (A1), (A2), (A3), (A4) *and* (A5) *are satisfied. Then the operator*

$$A_{\Phi,G,V} = \Delta - \nabla\Phi\cdot\nabla + G\cdot\nabla - V$$

with domain $D(A_{\Phi,G,V}) = W^{2,p}_V(\mathbb{R}^N,\mu)$ *generates an analytic semigroup* $T(\cdot)$ *on* $L^p(\mathbb{R}^N,\mu)$, $1<p<\infty$, *such that* $\|T(t)\|_{L^p(\mathbb{R}^N,\mu)}\le e^{t\beta/p}$, *where* $\mu(dx)=e^{-\Phi(x)}dx$. *Furthermore,* μ *is an invariant measure of* $T(\cdot)$, *that is,*

$$\int_{\mathbb{R}^N} T(t)f\mu(dx) = \int_{\mathbb{R}^N} f\mu(dx) \qquad \text{for } f\in C_b(\mathbb{R}^N),\ t\ge 0,$$

if and only if (A3)′ *holds in addition, where* $C_b(\mathbb{R}^N)$ *is the space of bounded continuous functions. Moreover,* $T(t)$ *is symmetric if* $p=2$ *and* $G=0$.

This theorem generalizes the result for the case where $V\equiv 0$ obtained by [1].

References

[1] G. Metafune, J. Prüss, A. Rhandi, and R. Schnaubelt, *L^p-regularity for elliptic operators with unbounded coefficients*, Adv. Differential Equations **10** (2005), 1131–1164.