Asymptotic energy concentration in the phase space of the weak solutions to the Navier-Stokes equations

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We consider the asymptotic behavior of the energy of weak solutions to the Navier-Stokes equations in $\mathbb{R}^n$, $n \geq 2$;

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + (u \cdot \nabla) u + \nabla p = 0 & \text{in } \mathbb{R}^n \times (0, \infty), \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}^n \times (0, \infty), \\ u(\cdot, 0) = a & \text{in } \mathbb{R}^n, \end{cases} \tag{N-S}$$

where $u = u(x,t) = (u_1(x,t), \dots, u_n(x,t))$ and $p = p(x,t)$ denote the unknown velocity vector and the pressure of the fluid at point $(x,t) \in \mathbb{R}^n \times (0, \infty)$, while $a = a(x) = (a_1(x), \dots, a_n(x))$ is a given initial velocity vector field.

Recently, another aspect of asymptotic behavior of the energy of solutions has been investigated. Skalák [4], [3] proved the asymptotic energy concentration in the following sense:

$$\lim_{t \to \infty} \frac{\|E_\lambda u(t)\|_2}{\|u(t)\|_2} = 1 \tag{1}$$

under the assumption that $\limsup_{t \to \infty} \|A^{1/2} u(t)\|_2 / \|u(t)\|_2 < \infty$ for the strong solution of (N-S), where $\{E_\lambda\}_{\lambda \geq 0}$ is the spectral decomposition of the Stokes operator A, and $\| \cdot \|_2$ denotes the L^2-norm. We say the energy of u concentrates in the phase (frequency) λ if u satisfies (1).

Our aim is to characterize the set of initial data which causes (1). For that purpose, we introduce the set

$$K_{m,\alpha}^{\delta} = \{\phi \in L^2\,;\, |\hat{\phi}(\xi)| \geq \alpha |\xi|^m \quad \text{for } |\xi| \leq \delta\} \tag{2}$$

for $\alpha, \delta > 0$ and $m \geq 0$. The set $K_{m,\alpha}^{\delta}$ may be regarded as a generalization of the set which was originally given by Schonbeck [2].

Theorem. *Let $2 \leq n \leq 4$, and let $r > 1$ and $m \geq 0$ be*

(i) *for $n = 2$,*

$$1 < r < \frac{4}{3}, \quad 0 \leq m < \frac{4}{r} - 3,$$

(ii) *for $n = 3, 4$,*

$$1 < r < \frac{n}{n-1}, \quad 0 \leq m < \frac{n}{r} - (n-1).$$

If $a \in L_\sigma^r \cap L_\sigma^2 \cap K_{m,\alpha}^{\delta}$ for some $\alpha, \delta > 0$, then for every turbulent solution $u(t)$ of (N-S) there exist $T > 0$ and a constant $C(n, r, m, \delta, \alpha, T) > 0$ such that

$$\left| \frac{\|E_\lambda u(t)\|_2}{\|u(t)\|_2} - 1 \right| \leq \frac{C}{\lambda} t^{-(n/r-n+1-m)} \tag{3}$$

holds for all λ and for all $t > T$.

Remark. Skalák [4] proved an energy concentration (1) under the assumption $\limsup_{t\to\infty} \|A^{1/2}u(t)\|_2/\|u(t)\|_2 < \infty$. From the assumption of Theorem 1, we can show that $\lim_{t\to\infty} \|A^{1/2}u(t)\|_2/\|u(t)\|_2 = 0$. On the other hand, our advantage seems to characterize the set of initial data which causes an energy concentration. Moreover, we get the explicit convergence rate of (1). The reason why we introduce the set $K_{m,\alpha}^{\delta}$ of initial data that causes (1) is because of the lower bound of the L^2-decay of solutions to (N-S).

To prove Theorem, the next Lemma plays an important role.

Lemma. *Let $2 \leq n \leq 4$. Let r and m be*

(i) *for $n = 2$,*

$$1 < r < \frac{4}{3}, \quad 0 \leq m < \frac{4}{r} - 3$$

(ii) *for* $n = 3, 4$,

$$1 < r < \frac{n}{n-1}, \quad 0 \leq m < \frac{n}{r} - (n-1).$$

If $a \in L^r_\sigma \cap L^2_\sigma \cap K^\delta_{0,\sigma}$, *Then every turbulent solution* $u(t)$ *of (N-S) satisfies*

$$\frac{\|\nabla u(t)\|_2^2}{\|u(t)\|_2^2} \leq O(t^{-(n/r-n+1-m)}), \tag{4}$$

as $t \to \infty$.

References

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