Holomorphic families of linear m-accretive operators in Banach spaces and application to Schrödinger operators in L^p

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For two closed linear operators T and A in a Banach space X we consider

$$T + \kappa A \text{ with domain } D_0 := D(T) \cap D(A),$$

where κ is a complex parameter and D_0 is assumed to be non-trivial.

Definition. $\{T(\kappa); \kappa \in G_0 \subset \mathbb{C}\}$ is said to be a *holomorphic family of type* (A) if

(**i**) $T(\kappa)$ is a closed linear operator with domain $D(T(\kappa)) = D$ independent of κ.

(**ii**) $T(\kappa)u$ is holomorphic with respect to $\kappa \in G_0$ for every $u \in D$.

Now we consider the Schrödinger type operators $-\Delta + \kappa V(x)$ with $V(x) \geq 0$. Here $T := -\Delta$ with domain $D(T) := W^{2,p}(\mathbb{R}^N)$ is *m-accretive* in $L^p = L^p(\mathbb{R}^N)$ $(1 < p < \infty$, $N \in \mathbb{N})$. Let A be the maximal operator of multiplication by $V(x) \in L^p_{\mathrm{loc}}(\mathbb{R}^N)$ (or $V(x) \in L^p_{\mathrm{loc}}(\mathbb{R}^N \setminus \{0\})$): $Au(x) := V(x)u(x)$ with domain $D(A) := \{u \in L^p; Vu \in L^p\}$. Then A is also m-accretive in L^p and its Yosida approximation is given by $A_\varepsilon v(x) = V_\varepsilon(x)v(x)$ $(v \in L^p)$, where $V_\varepsilon(x) := V(x)[1 + \varepsilon V(x)]^{-1}$ $\forall \varepsilon > 0$.

The nonnegative potential $V(x)$ is assumed to satisfy either (**V**) or $(\mathbf{V})_\varepsilon$:

(**V**) $V \in C^1(\mathbb{R}^N)$ and there are nonnegative constants a, b and c such that

$$|\nabla V(x)|^2 \leq a[V(x)]^3 + b[V(x)]^2 + c[V(x)] \quad \forall x \in \mathbb{R}^N. \tag{1}$$

$(\mathbf{V})_\varepsilon$ is nothing but condition (**V**) in which $V(x)$ is replaced with $V_\varepsilon(x)$.

Then, generalizing Kato [**1**, Theorem 7.1], we have the following

Theorem. *Let T and A be as stated above. Assume that* (**V**) (*or* $(\mathbf{V})_\varepsilon$). *Then*

(**i**) $\{T + \kappa A; \kappa \notin \Omega\} = \{-\Delta + \kappa V(x); \kappa \notin \Omega\}$ *forms a holomorphic family of type* (A),

(**ii**) $T + \kappa A = -\Delta + \kappa V(x)$ *is m-accretive in L^p for $\kappa \notin \Omega$ with* $\mathrm{Re}\,\kappa \geq 0$*, where*

$$\Omega : y^2 \leq \frac{p^2}{2(p-1)}\Big(x - \frac{p-1}{4}a\Big)\Big(\frac{(p-2)^2}{2(p-1)}x - \frac{p^2}{8}a\Big) \text{ and } x \leq \frac{p-1}{4}a \quad (x + iy \in \mathbb{C}). \tag{2}$$

Example. We consider several typical examples.

(**i**) Let $V(x) := |x|^2$. Then $a = b = 0$ and $c = 4$ in (1). Thus we see from Theorem that $\{-\Delta + \kappa|x|^2; \kappa \notin \Omega\}$ forms a holomorphic family of type (A) and $-\Delta + \kappa|x|^2$ is m-accretive in L^p for $\kappa \notin \Omega$ with $\mathrm{Re}\,\kappa \geq 0$, where Ω is given by a sector region of the complex plane.

(**ii**) Let $V(x) := |x|^{-2}$ and so $V_\varepsilon(x) = (|x|^2 + \varepsilon)^{-1}$. Then $a = 4$ and $b = c = 0$ in $(\mathbf{V})_\varepsilon$. Thus we see from Theorem that $\{-\Delta + \kappa|x|^{-2}; \kappa \notin \Omega\}$ forms a holomorphic family of type (A) and $-\Delta + \kappa|x|^{-2}$ is m-accretive in L^p for $\kappa \notin \Omega$ with $\mathrm{Re}\,\kappa \geq 0$, where Ω is given by a hyperbolic region of the complex plane. However, this result is not sharp. We can obtain a sharper result for the inverse square potential $|x|^{-2}$ which include Kato's result (see [**1**, Example 7.4]). Roughly speaking, instead of (2) we can obtain the closed hyperbolic region $y^2 \leq (x - C_1(p, N))(C_3(p)x - C_2(p, N))$ and $x \leq C_1(p, N)$ $(x + iy \in \mathbb{C})$, where $C_j(p, N)$ $(j = 1, 2)$ are constants dependent on p and N, and $C_3(p) \geq 0$ is a constant dependent on p.

References

[1] T. Kato, Remarks on holomorphic families of Schrödinger and Dirac operators, Differential Equations, Mathematics Studies **92**, North-Holland, Amsterdam, 1984, pp. 341–352.