Linear Schrödinger evolution equations with Coulomb potential with moving center[1]

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1. Problem

For $T > 0$ and $N \geq 3$ we consider the Cauchy problem for the Schrödinger equation:

$$\text{(SE)} \quad \begin{cases} i\dfrac{\partial u}{\partial t} + \Delta u + \dfrac{u}{|x - a(t)|} + V_1(x,t)u = f(x,t), \quad (x,t) \in \mathbb{R}^N \times [0,T], \\ u(\cdot, 0) = u_0(\cdot) \in H^2(\mathbb{R}^N) \cap H_2(\mathbb{R}^N) \end{cases}$$

in $L^2 = L^2(\mathbb{R}^N)$, where $a : [0,T] \to \mathbb{R}^N$ expresses the center of the Coulomb potential, V_1 and $f : \mathbb{R}^N \times [0,T] \to \mathbb{R}$ are another real-valued potential and an inhomogeneous term while

$$H_2(\mathbb{R}^N) := \{u \in L^2(\mathbb{R}^N); |x|^2 u \in L^2(\mathbb{R}^N)\}.$$

The existence of strong solutions to (SE) (with $f \equiv 0$) has been solved by Baudouin-Kavian-Puel [**1**] partly with *formal computation.* In this talk we reconstruct their argument with *rigorous proofs.*

2. Known result

Baudouin-Kavian-Puel established the case where $N = 3$ in the following

Theorem 1 ([**1**, Theorems 1 and 2]). *Assume that a and V_1 satisfy*

$$(1) \qquad \begin{cases} a \in W^{2,1}(0,T) := W^{2,1}(0,T;\mathbb{R}^N), \\ (1+|x|^2)^{-1}V_1 \in W^{1,1}\big(0,T; L^\infty(\mathbb{R}^N)\big) \quad \textit{and} \\ (1+|x|^2)^{-1}\nabla V_1 \in L^1\big(0,T; L^\infty(\mathbb{R}^N)\big)^N. \end{cases}$$

Then (SE) *with $f = 0$ has a unique solution u such that*

$$(2) \qquad \begin{aligned} u \in\, & W^{1,\infty}\big(0,T;L^2(\mathbb{R}^N)\big) \cap C_{\mathrm{w}}\big([0,T]; H^2(\mathbb{R}^N) \cap H_2(\mathbb{R}^N)\big), \\ & \cap L^\infty(0,T; H^2(\mathbb{R}^N) \cap H_2(\mathbb{R}^N)) \cap C([0,T]; H^1(\mathbb{R}^N) \cap H_1(\mathbb{R}^N)), \end{aligned}$$

where $u \in C_{\mathrm{w}}(I;H)$ means that u is weakly *continuous from I into H.*

First, we review their proof. For $\varepsilon > 0$ they consider the approximate problem

$$\text{(SE)}_\varepsilon \quad \begin{cases} i\dfrac{\partial u_\varepsilon}{\partial t} + \Delta u_\varepsilon + V_0^\varepsilon u_\varepsilon + V_1^\varepsilon u_\varepsilon = 0, \quad (x,t) \in \mathbb{R}^N \times [0,T], \\ u_\varepsilon(\cdot, 0) = u_0(\cdot) \in H^2(\mathbb{R}^N) \cap H_2(\mathbb{R}^N) \end{cases}$$

[1] This talk is based on my joint work with Professors N. Okazawa and T. Yokota.

and obtain $u_\varepsilon \in C([0,T];\ H^2(\mathbb{R}^N) \cap H_2(\mathbb{R}^N))$. Here the approximate potentials V_0^ε and $V_1^\varepsilon \in C([0,T]; C_b^2(\mathbb{R}^N))$ are defined as

$$\begin{aligned} V_0^\varepsilon(x,t) &:= (\varepsilon^2 + |x-a(t)|^2)^{-1/2}, \\ V_1^\varepsilon(x,t) &:= T_\varepsilon(V_1) * \zeta_\varepsilon = \int_{\mathbb{R}}\int_{\mathbb{R}^N} (T_\varepsilon \circ V_1)(x-\varepsilon y, t-\varepsilon s)\chi(s)\rho(y)\,dy\,ds. \end{aligned}$$

Let $\chi \in C_0^\infty(\mathbb{R})$ and $\rho \in C_0^\infty(\mathbb{R}^N)$ be nonnegative functions such that

$$\int_{-\infty}^{\infty} \chi(t)\,dt = \int_{\mathbb{R}^N} \rho(x)\,dx = 1.$$

Then T_ε and ζ_ε in the definition of V_1^ε are defined as

$$T_\varepsilon(s) := \operatorname{sign}(s)\min\{|s|, \varepsilon^{-1}\}, \quad \zeta_\varepsilon(x,t) := \varepsilon^{-1-N}\chi(t/\varepsilon)\rho(x/\varepsilon).$$

Among others they have shown the following energy estimates:

$$\|u_\varepsilon(t)\|_{H^2\cap H_2} + \left\|\frac{\partial u_\varepsilon}{\partial t}(t)\right\|_{L^2} \le C\|u_0\|_{H^2\cap H_2}, \quad t \in [0,T],$$

where C is independent of ε. Thus one can extract a subsequence $(u_{\varepsilon'})$, which converges to a solution of (SE) satisfying (2).

However, it seems that their proof has some errors. For instance, the definition of V_1^ε should be modified.

3. Our result

In this context the purpose of this talk is to rewrite the original proof in [**1**] correctly and to establish Theorem 1 with an inhomogeneous term.

Theorem 2. *In addition to* (1) *assume that* f *satisfies*

$$f \in W^{1,1}(0,T; L^2(\mathbb{R}^N)) \cap L^1(0,T; H^1(\mathbb{R}^N) \cap H_2(\mathbb{R}^N)). \tag{3}$$

Then (SE) *with initial value* $u_0 \in H^2(\mathbb{R}^N) \cap H_2(\mathbb{R}^N)$ *has a unique solution satisfying* (2) *and the energy estimate*:

$$\|\partial_t u(t)\| + \|u(t)\|_{H^2\cap H_2} \le C_0(\|u_0\|_{H^2\cap H_2} + \|f\|_F), \tag{4}$$

where $C_0 = C_0(a, V_1, T)$ *is a positive constant, while* $\|f\|_F$ *is given as follows*:

$$\|f\|_F := \|f\|_{L^\infty(0,T;\,L^2)} + \int_0^T (\|\partial_t f(t)\| + \alpha_1\|f(t)\|_{H^1} + \|f(t)\|_{H_2})\,dt;$$

$\alpha_1 > 0$ *is a constant depending on* a.

To prove Theorem 2 we define V_0^ε, V_1^ε and f more carefully. We define the extension operator $P : W^{1,1}(0,T;X) \to W^{1,1}(\mathbb{R};X)$ by

$$(P\varphi)(t) := \begin{cases} \varphi(t), & t \in [0,T], \\ (2 - t/T)\varphi(2T - t), & t \in (T, 2T], \\ (1 + t/T)\varphi(-t), & t \in [-T, 0), \\ 0, & \text{otherwise.} \end{cases}$$

Assume that $\operatorname{supp}\chi \subset [-1,1]$ and $\operatorname{supp}\rho \subset \overline{B(0;1)} := \{x \in \mathbb{R}^N; |x| \le 1\}$. Let $0 \le \eta \in W^{1,\infty}(0,\infty)$ be defined as

$$\eta(r) := \begin{cases} 1, & r \in [0,1), \\ 2 - r, & r \in [1,2), \\ 0, & r \in [2,\infty). \end{cases}$$

For $\varepsilon > 0$ let $\chi_\varepsilon(t) := \varepsilon^{-1}\chi(t/\varepsilon)$ and $\eta_\varepsilon(x) := \eta(\varepsilon|x|)$. Then we can define as

$$V_0^\varepsilon(x,t) := (\varepsilon^2 + |x - a_\varepsilon(t)|^2)^{-1/2}, \tag{5}$$

$$\begin{aligned} V_1^\varepsilon(x,t) &:= \big((\eta_\varepsilon(PV_1)) * \zeta_\varepsilon\big)(x,t) \\ &= \int_{B(0;1)} \Big[\int_{-1}^{1} \eta_\varepsilon(x - \varepsilon y)(PV_1)(x - \varepsilon y, t - \varepsilon s)\chi(s)\rho(y)\, ds\Big]\, dy, \end{aligned} \tag{6}$$

$$\begin{aligned} f_\varepsilon(x,t) &:= \big((Pf) * \zeta_\varepsilon\big)(x,t) \\ &= \int_{B(0;1)} \Big[\int_{-1}^{1} (Pf)(x - \varepsilon y, t - \varepsilon s)\chi(s)\rho(y)\, ds\Big]\, dy. \end{aligned} \tag{7}$$

In (5) a_ε is defined as

$$a_\varepsilon(t) := a(0) + \int_0^t \Big(\Big(P\frac{da}{ds}\Big) * \chi_\varepsilon\Big)(s)\, ds.$$

Then $(\mathrm{SE})_\varepsilon$ is dealt with by [**2**, Theorem 1.4]. In a similar way as in [**1**] we prove that the family $\{u_\varepsilon\}$ of solutions to $(\mathrm{SE})_\varepsilon$ satisfies

$$\|\partial_t u_\varepsilon(t)\| + \|u_\varepsilon(t)\|_{H^2 \cap H_2} \le C_0(\|u_0\|_{H^2 \cap H_2} + \|f\|_F)$$

and there exists a unique strong solution u of (SE) satisfying (2) and (4).

References

[1] L. Baudouin, O. Kavian and J.-P. Puel, *Regularity for a Schrödinger equation with singular potentials and application to bilinear optimal control*, J. Differential Equations **216** (2005), 188–222.

[2] N. Okazawa, *Remarks on linear evolution equations of hyperbolic type in Hilbert space*, Adv. Math. Sci. Appl. **8** (1998), 399–423.