Kato-smooth operators and wave operators for Schrodinger equaitons on scattering manifolds

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1 Introduction

In this seminar, we study a class of self-adjoint second-order elliptic operators, which includes Laplacians on noncompact manifolds which are polynomially large at infinity. Proving the Mourre estimate and applying the Mourre theory to these operators, we show that there is no accumulation of embedded eigenvalues inside the essential spectrum. We also obtain resolvent estimates which show the absence of singular spectrum. Using the resolvent estimates, we obtain three types of Kato-smooth operators.

We also construct a time-dependent scattering theory. Consider two operators in our class. If the perturbation is "short-range", it admits a factorization into a product of Kato-smooth operators. The smooth-method of Kato proves the existence and completeness of wave operators.

2 Settings

Let M be an n-dimensional smooth non-compact manifold such that $M = M_C \cup M_\infty$, where M_C is compact and M_∞ is the noncompact end. We assume M_∞ has the form $\mathbb{R}_+ \times N$ with N compact. We suppose that a positive C^∞ density ω is given on M which has the property on M_∞,

$$\omega = dr \cdot \mu$$

where r is a coordinate in $\mathbb{R}_+$ and μ is a smooth positive density on N. Then $\mathcal{H} = L^2(M,\omega)$ and $L^2(N,\mu)$ are well defined. We consider as our "free operator" L_0, self-adjoint second-order elliptic operators of the form

$$L_0 = D_r^2 + k(r)P$$

when acting on functions supported in $(1,\infty)\times N$. Here $D_r = i^{-1}\partial_r$, P is a self-adjoint second-order elliptic nonnegative operator acting on $L^2(N,\mu)$, and k is a positive smooth function of r such that for some $c, C > 0$,

$$\begin{aligned} cr^{-1}k \le -k' \le Cr^{-1}k \\ |k''| \le Ck. \end{aligned} \tag{1}$$

We will assume that L is a second-order elliptic operator on M, whose closure on C_0^∞ is self-adjoint, such that

$$L = L_0 + E,$$

where there are finitely many coordinate charts on M_∞ such that in coordinate $(r, \theta_1, \cdots, \theta_{n-1})$ on M_∞, E has the form

$$E = (1, D_r, \sqrt{k}\tilde{D}_\theta) \begin{pmatrix} V & b_1 & b_2 \\ b_1 & a_1 & a_2 \\ {}^t b_2 & {}^t a_2 & a_3 \end{pmatrix} \begin{pmatrix} 1 \\ D_r \\ \sqrt{k}\tilde{D}_\theta \end{pmatrix}$$

where $\tilde{D}_\theta = \mu(\theta)^{-\frac{1}{2}} D_\theta \mu_\theta^{\frac{1}{2}}$, $\mu(\theta)$ is defined by $\mu = \mu(\theta) d_{\theta_1} \cdots d_{\theta_{n-1}}$. The coefficients a_1, a_2, b_1, b_2, and V have support in M_∞ and are smooth real-valued functions of$(r, \theta_1, \cdots, \theta_{n-1})$ such that

$$|\partial_r^l \partial_\theta^\alpha e(r,\theta)| \leq C_{l,\alpha} r^{-\nu-l}, \quad \nu > 0 \tag{2}$$

where $e = a_1, a_2, b_1, b_2$, or V.

Remark 1. *The case where M is a Riemannian manifold, the metric on M_∞ is "close" to a warped product of $\mathbb{R}_+$ and N, and L is the Laplace operator, fits into the framework described above. The function$\sqrt{k(r)}$, varies inversely with the size of $M_\infty = \mathbb{R}_+ \times N$. A typical exapmle of k which satisfies* (1) *is given by $k(r) = r^{-\alpha}, \alpha > 0$. The case $\alpha = 2$ corresponds to scattering manifolds including asymptotically Euclidean spaces.*

Let $\chi(r) \in C(\mathbb{R})$ be real-valued such that $\chi(r) = 1$ if $r \geq 1$ and $\chi(r) = 0$ if $r \leq \frac{1}{2}$. Let $\chi_R(r) = \chi(\frac{r}{R})$. Set

$$A = \frac{1}{2}(\chi_R^2 r D_r + D_r r \chi_R^2).$$

3 Main Results

Theorem 2. *Suppose $L = L_0 + E$, where k satisfies* (1) *and the coefficients in E obey the bounds* (2) *with $\nu > 0$. Then $\sigma_{ess}(L) = \mathbb{R}_+ \cup \{0\}$ and L satisfies a Mourre estimate at each point in $\mathbb{R}_+$ with conjugate operator A. L and A satisfy hypotheses about boundedness of commutators to apply the Mourre theory. In particular, eigenvalues of L do not accumulate in $\mathbb{R}_+$, and $\sigma_{sc}(L) = \emptyset$. We also obtain the resolvent estimates:*

$$\sup_{z \in \Lambda_\pm = \Lambda \pm i\mathbb{R}_+} \|(|A|+1)^{-1}(L-z)^{-1}(|A|+1)^{-1}\| < \infty$$

if $\Lambda \Subset \mathbb{R} \setminus \sigma_{pp}(L)$.

Remark 3. *Theorem 2 is solved in [1] when $\nu = 2$.*

Theorem 4. *Under the hypothesis of Theorem 2, the operators*

$$\langle r \rangle^{-s}, \ \chi_R \langle r \rangle^{-s} D_r, \ \text{and} \ \ \chi_R \langle r \rangle^{-\frac{1}{2}} (kP)^{\frac{1}{2}}$$

are L-smooth on Λ if $\Lambda \Subset \mathbb{R} \setminus \sigma_{pp}(L)$ and $s > \frac{1}{2}$.

Theorem 5. *Suppose $\nu > 1$. Then the wave operators*

$$W^\pm(L, L_0) = s - \lim_{t \to \pm\infty} e^{itL} e^{-itL_0} P_{ac}(L_0)$$

and $W^\pm(L_0, L)$exist and are adjoint each other. They are complete and give the unitarily equivalence between $L_0^{(ac)}$ and $L^{(ac)}$.

Reference

[1] R. Froese, P. Hislop.: Spectral analysis of second-order elliptic operators on noncompact manifolds. Duke J. Math. (58) 1 (1989), 103-129.

[2] P. Perry, I. M. Sigal, B. Simon.: Spectral Analysis of N-body Schrödinger operators. Ann. Math. 114 (1981) 519-567.