Continuous dependence on modelling parameters
for the complex Ginzburg-Landau equation
with inhomogeneous boundary condition

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Let $\Omega$ be a star-shaped (with respect to the origin) bounded domain in $\mathbb{R}^d$ ($d \geq 2$) with smooth boundary $\partial \Omega$. Then we consider the following initial-boundary value problem for the complex Ginzburg-Landau equation with “inhomogeneous” Dirichlet boundary condition:

$$
\left\{
\begin{array}{ll}
\frac{\partial u}{\partial t} = \gamma u - (\mu + i\nu)|u|^2u + (\alpha + i\beta)\Delta_x u & \text{in } \Omega \times (0, T), \\
u = u_B & \text{on } \partial \Omega \times (0, T), \\
u(x, 0) = u_0(x), & \text{in } \Omega,
\end{array}
\right.
$$

where $i = \sqrt{-1}$, $\mu > 0$, $\alpha > 0$, $\gamma$, $\nu$, $\beta \in \mathbb{R}$, $\Delta_x u := \sum_{k=1}^d \partial^2 u / \partial x_k^2$ and $u$ is a complex-valued unknown function. Here $\Omega$ and $\mu + i\nu$ satisfy the following additional conditions:

(A1) $\min \{x \cdot n(x); x \in \partial \Omega\} > 0$, where $n(x)$ is the unit outward normal vector at $x \in \partial \Omega$;

(A2) $|\nu| \leq \sqrt{3}\mu$,

where (A2) is determined by the Liskevich-Perelman inequality (see e.g., [1, Lemma 1.2]).

Yang-Gao [2] showed that the solution to (CGL)$_{\mu, \nu}$ depends continuously on the modelling parameter $\mu + i\nu$ under the various cases. Their results and ours are summarized as follows.

<table>
<thead>
<tr>
<th>[2, Th. 2.2]</th>
<th>[2, Th. 2.4]</th>
<th>[2, Th. 3.1]</th>
<th>Main Theorem</th>
</tr>
</thead>
<tbody>
<tr>
<td>spatial dimension</td>
<td>$2 \leq d \leq 4$</td>
<td>$d \geq 2$</td>
<td>$2 \leq d \leq 6$</td>
</tr>
<tr>
<td>initial value</td>
<td>$u_0 \in L^2(\Omega)$</td>
<td>$u_0 \in L^4(\Omega)$</td>
<td>$u_0 \in L^2(\Omega)$</td>
</tr>
<tr>
<td>boundary value</td>
<td>$u_B \in H^2$</td>
<td>$</td>
<td>u_B</td>
</tr>
<tr>
<td>restriction on $\alpha, \beta$</td>
<td>nothing</td>
<td>$</td>
<td>\beta</td>
</tr>
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The purpose of this talk is to establish the following theorem which improves all the three theorems [2, Theorems 2.2, 2.4 and 3.1].

**Main Theorem.** Let $\Omega$ satisfy (A1) and let $\mu_j, \nu_j$ satisfy (A2), i.e., $|\nu_j| \leq \sqrt{3}\mu_j$ ($j = 1, 2$). Assume that $d \geq 2$, $u_0 \in L^2(\Omega)$ and $u_B \in W^{2,4}(\partial \Omega \times (0, T))$. Let $u_j$ be a solution to (CGL)$_{\mu_j, \nu_j}$ ($j = 1, 2$). Then the solution depends continuously on the modelling parameter $\mu + i\nu$, i.e.,

$$
\|u_1(t) - u_2(t)\|^2_{L^2(\Omega)} \leq 4\sqrt{(\mu_1 - \mu_2)^2 + (\nu_1 - \nu_2)^2} (1 + \mu_1^{-1} + \mu_2^{-1})C^2(t),
$$

where $C_1 = C_1(\gamma, \alpha, |\Omega|)$ and $C_2(t) = C_2(t, \alpha, \beta, \gamma, \|u_B\|_{W^{2,4}(\partial \Omega \times (0, t))}, \|u_0\|_{L^2(\Omega)})$ are constants.

The following proposition plays an essential role in the proof of Main Theorem.

**Proposition.** Assume (A1). Let $u_B \in W^{2,4}(\partial \Omega \times (0, T))$. Then there exists the auxiliary function $\psi$ such that $\psi(t) \in W^{2,4}(\Omega)$ a.a. $t \in (0, T)$, $\psi \in L^4(\Omega \times (0, T)) \cap H^1(\Omega \times (0, T))$ and

$$
\left\{
\begin{array}{ll}
\Delta_x \psi = 0 & \text{in } \Omega \times (0, T), \\
\psi = u_B & \text{on } \partial \Omega \times (0, T).
\end{array}
\right.
$$

**References**
