On the standing waves of negative Hartree equation

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In this talk, we consider the following nonlinear Schrödinger equation with nonlocal nonlinearity which increases at the spatial infinity.

$$2iu_t = -\Delta u + \lambda |x|^2 u + \eta \left(|x|^\gamma * |u|^2\right) u, \ (t,x) \in \mathbb{R}^{1+d}, \tag{NH}$$

where, $\lambda, \eta \in \mathbb{R}$ and $|x|^\gamma * |u|^2 := \int_{\mathbb{R}^d} |x-y|^\gamma |u(y)|^2 \, dy$. This equation can be considered as a generalization of the Schrödinger-Poisson (or Schrödinger-Newton) system in low dimensions. Indeed, if $d = 1$ and $\gamma = 1$, (NH) can be rewritten as

$$\begin{cases} i\partial_t u = -\Delta u + \lambda |x|^2 u + \eta P u, \\ \Delta P = |u|^2. \end{cases} \tag{SP}$$

In this talk, we consider the case $\gamma = 2$.

The time global existence of (NH) was shown by Masaki [4].

Theorem 1 ([4], Theorem 2.1)**.** *Let $\gamma = 2$. Then (NH) is globally well-posed in $C(\mathbb{R}; H^{1,1})$, where,*

$$H^{1,1} := \left\{ \phi \in H^1 \mid \langle x \rangle \phi \in L^2 \right\}.$$

Remark 1. *Actually, we can show the global well-posedness in*

$$H^{1/2,1/2} := \left\{ \phi \in H^{1/2} \mid \langle x \rangle^{1/2} \phi \in L^2 \right\},$$

by modifying the definition of the solution.

A standing wave is a solution of (NH) with the form $e^{-i\omega t/2}\phi(x)$.

Definition 1. *Let $e^{-i\omega t/2}\phi(x)$ be the solution of (NH). We say $e^{-i\omega t/2}\phi(x)$ is stable if for all $\varepsilon > 0$, there exists $\delta > 0$ which satisfies the following. If $||u_0 - \phi||_{H^{1,1}} < \delta$, then*

$$\sup_{t>0} \inf_s ||e^{is}\phi - u(t)||_{H^{1,1}} < \varepsilon,$$

where $u(t)$ is the solution of (NH) and satisfies $u(0) = u_0$. If $e^{-i\omega t/2}\phi(x)$ is not stable, then we say it is unstable.

To study the stability of standing waves, the theory of Grillakis-Shatah-Strauss [1, 2] is useful. Let E be a C^2 functional and E satisfies $E(e^{is}u) = E(u)$ for all $s \in \mathbb{R}$. We consider the following Hamiltonian PDE.

$$2iu_t = E'(u), \tag{1}$$

This equation formally conserves L^2 norm and the energy E. Further, (1) is a generalization of (NH). Let $S_\omega(u) = E(u) - \frac{1}{2}\omega ||u||_{L^2}^2$. Then, we have $S'_\omega(\phi_\omega) = 0$ if and only if $e^{-i\omega t/2}\phi_\omega$ is a solution.

Theorem 2 ([1, 2])**.** *Let $n(S''_\omega(\phi_\omega))$ be the number of negative eigenvalues (Morse index) of $S''_\omega(\phi_\omega)$. Further, suppose the kernel of $S''_\omega(\phi_\omega)$ is spanned by $i\phi_\omega$. Set $d(\omega) := S_\omega(\phi_\omega)$. Then, we have the following.*

(i) *If $n(S''_\omega(\phi_\omega)) = 0$ or if $n(S''_\omega(\phi_\omega)) = 1$ and $d''(\omega) > 0$, then $e^{-i\omega t/2}\phi(x)$ is stable.*

(ii) *If $d''(\omega) \neq 0$ and $n(S''_\omega(\phi_\omega)) - \max(d''(\omega)/|d''(\omega)|, 0)$ is odd, then $e^{-i\omega t/2}\phi(x)$ is unstable.*

If $n(S''_\omega(\phi_\omega)) - \max(d''(\omega)/|d''(\omega)|, 0) \geq 2$ and even, we cannot use the Grillakis-Shatah-Strauss theorem. These cases are the case which the standing waves have high Morse index (and high energy). So the standing waves are expected to be unstable. However, in contrast to gradient system, notice that we cannot directly obtain the instability of standing waves from the high Morse index condition. Indeed, the instability problem of standing waves with high Morse index are widely open.

Here, we give a counter example to the vague conjecture that all standing waves with high Morse index is unstable.

Theorem 3 (Maeda-Masaki [3])**.** (i) *Let $d = 1$, $\lambda = 0$ and $\eta = 1$. Then for all $m \in \mathbb{N}$, there exists a stable standing wave $e^{-i\omega t/2}\phi_{\omega,m}$ with $d''(\omega) < 0$ and $n(S''_\omega(\phi_\omega)) = 2m$.*

(ii) *Let $d = 1$, $\lambda = 2$ and $\eta = -1$. Then for all $m \in \mathbb{N}$, there exists a stable standing wave $e^{-i\omega t/2}\phi_{\omega,m}$ with $d''(\omega) > 0$ and $n(S''_\omega(\phi_\omega)) = 2m + 1$.*

Remark 2. *The standing waves in (i), (ii) can be explicitly written as*

$$\begin{aligned} Q_{i,n}(t,x;M) &:= \exp\left(-\frac{3}{2}M^{1/2}\left(n+\frac{1}{2}\right)t\right) M^{\frac{1}{2}+\frac{1}{8}}\Omega_n(M^{1/4}x), \\ Q_{ii,n}(t,x;M) &:= \exp\left(-\left(n+\frac{1}{2}\right)\left(\kappa^{1/2}-\frac{1}{2}M\kappa^{-1/2}\right)t\right) M^{\frac{1}{2}}\kappa^{\frac{1}{8}}\Omega_n(\kappa^{1/4}x). \end{aligned}$$

Here, $\kappa = 2 - M$ and Ω_n satisfies the following.

$$\frac{1}{2}\left(-\Delta + |x|^2\right)\Omega_n = \left(n+\frac{1}{2}\right)\Omega_n, \ \ ||\Omega_n||_2 = 1.$$

The theorem can be shown by computing the solution formula given by Masaki [4].

Theorem 4 ([4], Theorem 2.1)**.** *Let*

$$X[u] := \int_{\mathbb{R}^d} x|u|^2\,dx, \qquad P[u] := \operatorname{Im}\int_{\mathbb{R}^d} \bar{u}\nabla u\,dx = \int_{\mathbb{R}^d} \xi|\mathcal{F}u|^2\,d\xi.$$

We also introduce energy

$$E[u] = \frac{1}{2}\|\nabla u\|_{L^2}^2 + \frac{\lambda}{2}\|xu\|_{L^2}^2 + \frac{\eta}{4}\iint_{\mathbb{R}^{d+d}} |x-y|^2|u(x)|^2|u(y)|^2dxdy.$$

Let $M = ||u_0||_2^2$, $\mathbf{a} = P[u_0]/M$, $\mathbf{b} = X[u_0]/M$, and $\kappa = \lambda + \eta M$. Note that mass $M[u(t)] = \|u\|_{L^2}^2$ and energy $E[u(t)]$ are conserved. Then the solution is written as

$$\begin{aligned} u(t) &= e^{-i\Psi(t)}\mathcal{G}_\lambda(t,\mathbf{a},\mathbf{b})\mathcal{U}_\kappa(t)\mathcal{G}_\kappa(0,\mathbf{a},\mathbf{b})^{-1}u_0, \\ &= e^{-i\Psi(t)}\mathcal{G}_\lambda(t,\mathbf{a},\mathbf{b})\mathcal{G}_\kappa(t,\mathbf{a},\mathbf{b})^{-1}\mathcal{U}_\kappa(t)u_0, \end{aligned} \tag{2}$$

where $\Psi(t)$ *is given by*

$$\Psi(t) = \frac{\eta}{2} \int_0^t ||x\mathcal{U}_\kappa(s)\mathcal{G}_\kappa(0, \mathbf{a}, \mathbf{b})^{-1} u_0||_2^2 \, ds,$$

$\mathcal{U}_\kappa$ *is given by*

$$\mathcal{U}_\kappa(t) = e^{i\frac{t}{2}(\Delta - \kappa |x|^2)},$$

and $\mathcal{G}_\kappa$ *is given by*

$$(\mathcal{G}_\kappa(t, \mathbf{a}, \mathbf{b})\phi)(x) = e^{-\frac{i}{2} g_\kappa(t,\mathbf{a},\mathbf{b}) \cdot g'_\kappa(t,\mathbf{a},\mathbf{b})} e^{ix \cdot g'_\kappa(t,\mathbf{a},\mathbf{b})} \phi(x - g_\kappa(t, \mathbf{a}, \mathbf{b})). \tag{3}$$

Here, $g_\kappa(t, \mathbf{a}, \mathbf{b})$ *is a solution of an ODE* $g''_\kappa = -\kappa g_\kappa$*,* $g'_\kappa(0) = \mathbf{a}$*, and* $g_\kappa(0) = \mathbf{b}$.

References

[1] Manoussos Grillakis, Jalal Shatah, and Walter Strauss, *Stability theory of solitary waves in the presence of symmetry. I*, J. Funct. Anal. **74** (1987), no. 1, 160–197. MR MR901236 (88g:35169)

[2] ———, *Stability theory of solitary waves in the presence of symmetry. II*, J. Funct. Anal. **94** (1990), no. 2, 308–348. MR MR1081647 (92a:35135)

[3] Masaya Maeda and Satoshi Masaki, *An example of stable excited state on nonlinear Schrödinger equation with nonlocal nonlinearity*, archived as `arXiv:1109.2653`, 2011.

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