Heat asymptotics for Dirichlet elliptic operators with non-smooth coefficients

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Let us consider a $2m$th-order strongly elliptic operator in divergence form

$$Au(x) = \sum_{|\alpha|\le m,\, |\beta|\le m} D^\alpha(a_{\alpha\beta}(x) D^\beta u(x)) \tag{1}$$

defined in a domain Ω of $\mathbb{R}^n$, and assume that $a_{\alpha\beta}(x) = \overline{a_{\beta\alpha}(x)} \in L_\infty(\mathbb{R}^n)$. We denote by $A_{L_2(\Omega)}$ the self-adjoint operator in $L_2(\Omega)$ associated with A and the Dirichlet boundary conditions.

When Ω is a bounded domain, the spectrum of $A_{L_2(\Omega)}$ is descrete. Let λ_j $(j = 1, 2, \cdots)$ be the eigenvalues of $A_{L_2(\Omega)}$, and $\varphi_j(x)$ the corresponding normalized eigenfunction. Under the restricted smoothness conditions on the coefficients and the boundary of Ω, we consider the asymptotic behavior of the following functions at $x = y$ as $\lambda \to \infty$, or as $t \to +0$:

- the counting function $N(\lambda) = \sum_{\lambda_j \le \lambda} 1$.
- the spectral function $e(\lambda, x, y) =$ Kernel of the spectral projector $E(\lambda)$ of $A_{L_2(\Omega)}$.
  $e(\lambda, x, x) = \sum_{\lambda_j \le \lambda} |\varphi_j(x)|^2$ if Ω is bounded.
- the partition function $U(t) =$ Trace of $e^{-tA_{L_2(\Omega)}} = \sum_{j=1}^\infty e^{-t\lambda_j}$.
- $U(t, x, y) =$ Kernel of the semigroup $\exp(-tA_{L_2(\Omega)})$.
  $U(t, x, x) = \sum_{j=1}^\infty e^{-t\lambda_j} |\varphi_j(x)|^2$ if Ω is bounded.

We note that the relations

$$\begin{aligned} N(\lambda) &= \int_\Omega e(\lambda, x, x)\, dx, \\ U(t) &= \int_{-\infty}^\infty e^{-t\lambda}\, dN(\lambda) = \int_\Omega U(t, x, x)\, dx, \\ U(t, x, x) &= \int_{-\infty}^\infty e^{-t\lambda}\, d_\lambda e(\lambda, x, x) \end{aligned}$$

hold.

For $\rho > 0$ the space $C_L^\rho(\Omega)$ is defined by $C_L^\rho(\Omega) = C^{\rho_0,\rho_1}(\Omega)$ if ρ is written as $\rho = \rho_0 + \rho_1$ with $\rho_0 \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$ and $0 < \rho_1 \le 1$. We assume that there exists $\sigma > 0$ such that

$$a_{\alpha\beta} \in C_L^{\sigma-2m+|\alpha|+|\beta|}(\mathbb{R}^n) \quad \text{for } |\alpha| + |\beta| > 2m - \sigma.$$

Let

$$\delta(x) = \operatorname{dist}(x, \partial\Omega).$$

Theorem 1 ([4]). *Let $0 < \sigma \le 1$, and let Ω be a C^1 bounded domain. Then for any $\theta \in (0, \sigma)$*

$$e(\lambda^{2m}, x, x) = c_A(x)\lambda^n + O(\lambda^{n-\theta} + \delta(x)^{-1}\lambda^{n-1}) \tag{2}$$

as $\lambda \to \infty$, where O-estimate is uniform with respcet to $x \in \Omega$, and $c_A(x) = (2\pi)^{-n} \int_{a_0(x,\xi)<1} d\xi$. Here $a_0(x, \xi)$ denotes the principal symbol of A.

In addition,

$$N(\lambda^{2m}) = c_{A,\Omega}\lambda^n + O(\lambda^{n-\theta}) \tag{3}$$

as $\lambda \to \infty$, where $c_{A,\Omega} = (2\pi)^{-n} \int_\Omega dx \int_{a_0(x,\xi)<1} d\xi$.

Remark 2. *Zielinski [8] obtained* (3) *when* $2m > n$.

In order to characterize the coefficients in the asymptotic formula of $U(t, x, x)$ for the case of $\Omega = \mathbb{R}^n$ or $\Omega = \mathbb{R}^n$, we set

$$\mathbf{a}_0 = (a_{\alpha\beta})_{(\alpha,\beta)\in I_0}, \quad \mathbf{a}_1 = (a^{(\gamma)}_{\alpha\beta})_{(\alpha,\beta,\gamma)\in I_1}$$

with $I_0 = \{(\alpha, \beta) : |\alpha| = |\beta| = m\}$ and $I_1 = \{(\alpha, \beta, \gamma) : |\alpha| \le m,\ |\beta| \le m,\ \gamma \in \mathbb{N}_0^n\}$.

Definition 3. *For $j \in \mathbb{N}_0$ we say that a polynomial $F(\mathbf{a}_1)$ has weight j if $F(\mathbf{a}_1)$ is written as*

$$\sum const \times a^{(\gamma^1)}_{\alpha^1\beta^1} \cdots a^{(\gamma^k)}_{\alpha^k\beta^k},$$

where the sum is finite and $(\alpha^l, \beta^l, \gamma^l) \in I_1$ with $1 \le l \le k$ satisfy

$$\sum_{l=1}^{k} (2m - |\alpha^l| - |\beta^l| + |\gamma^l|) = j.$$

We also say that a constant has weight 0. *The set of polynomials of weight j is denoted by $\mathcal{P}_j$.*

We write $x \in \mathbb{R}^n_+$ as $x = (x', x_n)$ with $x_n > 0$.

Theorem 4 ([5])**.** *Let $\Omega = \mathbb{R}^n$. Then there exist functions $b_j(x) \in C^{\sigma-j}_L(\mathbb{R}^n)$ with $j \in \mathbb{N}_0$, $0 \le j < \sigma$ such that*

$$U(t^{2m}, x, x) = \sum_{0\le j<\sigma} b_j(x)t^{j-n} + O(t^{\sigma-n}) \quad as\ t \to +0,$$

where the O-estimate is uniform with respect to x.

Furthermore, the function $b_j(x)$ is written as the sum of terms of the form $F(\mathbf{a}_1(x))B(\mathbf{a}_0(x))$, where $F \in \mathcal{P}_j$ and B is a C^∞ function of $\mathbf{a}_0$ in a suitable open set O_A. In particular,

$$b_0(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \exp(-a_0(x, \xi))\, d\xi. \tag{4}$$

Theorem 5 ([5])**.** *Let $\Omega = \mathbb{R}^n_+$. Then there exist functions $b_j(x) \in C^{\sigma-j}_L(\mathbb{R}^n)$ and $w_j(x', s)$ with $j \in \mathbb{N}_0$, $0 \le j < \sigma$ such that $w_j(x', s)$ is Hölder continuous of exponent $\sigma - j$ in $x' \in \mathbb{R}^{n-1}$ and of class C^∞ in $s > 0$ and that*

$$U(t^{2m}, x, x) = \sum_{0\le j<\sigma} b_j(x)t^{j-n} - \sum_{0\le j<\sigma} w_j(x', t^{-1}x_n)t^{j-n} + O(t^{\sigma-n}) \quad as\ t \to +0,$$

where the O-estimate is uniform with respect to x.

Furthermore, $b_j(x)$ is the same as in Theorem 4, and the function $w_j(x', s)$ is written as the sum of terms of the form $F(\mathbf{a}_1(x', 0))W(\mathbf{a}_0(x', 0), s)$, where $F \in \mathcal{P}_j$ and W is a C^∞ function of $(\mathbf{a}_0, s)$ in $O_A \times \mathbb{R}_+$ satisfying the following properties:

(i) There exist $C = C(n, m, \sigma, A) > 0$ and $c = c(n, m, \sigma, A) > 0$ such that $|W(\mathbf{a}_0, s)| \le C \exp(-cs^{m_0})$ with $m_0 = 2m/(2m - 1)$;

(ii) For $l \in \mathbb{N}$, $\int_0^\infty s^{l-1}W(\mathbf{a}_0, s)\, ds$ is a C^∞ function of $\mathbf{a}_0$ in O_A and there exists $C = C(n, m, \sigma, l, A) > 0$ such that $\int_0^\infty s^{l-1}|W(\mathbf{a}_0, s)|\, ds \le C$.

Theorem 6 ([5])**.** *Assume that* Ω *is a bounded domain with* $C_L^{\sigma+1}$ *boundary. Then there exist constants* c_j *with* $0 \le j < \sigma$, $j \in \mathbb{N}_0$ *such that*

$$U(t^{2m}) = \sum_{0 \le j < \sigma} c_j t^{j-n} + O(t^{\sigma-n}) \quad \text{as } t \to +0. \tag{5}$$

In particular, $c_0 = \int_\Omega b_0(x)\,dx$ *with* $b_0(x)$ *given in* (4)*.*

If $0 < \sigma \le 1$, we can weaken the smoothness assumption on the boundary.

Theorem 7 ([5])**.** *Let* $0 < \sigma \le 1$ *and assume that* Ω *is a bounded domain with* C^1 *boundary. Then there exists a constant* $c = c(n, m, A, \Omega) > 0$ *such that*

$$U(t^{2m}, x, x) = b_0(x)t^{-n} + O(t^{\sigma-n} + t^{-n}\exp[-c(t^{-1}\delta(x))^{m_0}]) \quad \text{as } t \to +0 \tag{6}$$

with $b_0(x)$ *given in* (4)*, where the* O*-estimate is uniform with respect to* x*. Furthermore,*

$$U(t^{2m}) = c_0 t^{-n} + O(t^{\sigma-n}) \quad \text{as } t \to +0. \tag{7}$$

Remark 8. *Mizohata and Arima [6] obtained* (6) *with* $\sigma = 1$ *in the* C^∞ *settings under general boundary conditions.*

We would like to emphasize that the assumption $2m > n$ is not required. Our main tool to derive the above results is the L_p theory for divergence-form elliptic operators developed in [2, 3]. In order to prove Theorems 4, 5 we approximate A by an operator A^ε which has C^∞ coefficients and construct a parametrix for $\exp(-t(A^\varepsilon)_{L_2(\Omega)})$ more elaborately than Greiner [1] (see also [7]) did.

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