Life span of positive solutions for a semilinear heat equation with non-decaying initial data

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Abstract

We show an upper bound of the life span of positive solutions of a semilinear heat equation for non-decaying initial data. The bound is expressed by the limit inferior of the data at space infinity around a specific direction. We also show that the minimal time blow-up occurs when initial data attains its maximum at space infinity.

We study the life span of positive solutions of the Cauchy problem for a semilinear heat equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u + F(u), & (x,t) \in \mathbb{R}^n \times (0,\infty), \\ u(x,0) = \phi(x) \geq 0, & x \in \mathbb{R}^n, \end{cases} \tag{1}$$

where $n \geq 2$. Let ϕ be a bounded continuous function on $\mathbb{R}^n$. Throughout this talk, we assume that $F(u)$ satisfies

$$F(u) \geq u^p \quad \text{for } u \geq 0, \tag{2}$$

with $p > 1$.

In this talk, we show a upper bound on the life span of positive solutions of equation (1) for non-decaying initial data. We define the life span (or blow up time) T^* as

$$T^* = \sup\{T > 0 \mid (1) \text{ possesses a unique classical solution in } \mathbb{R}^n \times [0,T)\}. \tag{3}$$

In the case $F(u) = u^p$, results in [2, 7, 8, 17] are summarized as follows:
(i) Let $p \in (1, 1+2/n]$. Then every nontrivial solution of the equation (1) blows up in finite time.
(ii) Let $p \in (1+2/n, \infty)$. Then the equation (1) has a time-global classical solution for some initial data ϕ.

Especially for non-decaying initial data, it was shown that the solution of the equation (1) blows up in finite time for any $p > 1$. This result was proved in [9, 11].

Recently, several studies have been made on the life span of solutions for (1). See [1, 3-6, 9-19], and references therein. Gui and Wang [6] proved the following results when initial data takes the form $\phi(x) = \lambda\psi(x)$.
(i) $\lim_{\lambda\to\infty} T^* \cdot \lambda^{p-1} = \frac{1}{p-1}\|\psi\|_{L^\infty(\mathbb{R}^n)}^{1-p}$.
(ii) If $\lim_{|x|\to\infty} \psi(x) = k$, then $\lim_{\lambda\to 0} T^* \cdot \lambda^{p-1} = \frac{1}{p-1}k^{1-p}$.

The purpose of this talk is to give a upper bound of the life-span of the solution for the equation (1) with initial data having positive inferior limit at space infinity.

In order to state main results, we prepare several notations. For $\xi' \in \mathbb{S}^{n-1}$, and $\delta \in (0, \sqrt{2})$, we set neighborhood $S_{\xi'}(\delta)$:

$$S_{\xi'}(\delta) := \left\{ \eta' \in \mathbb{S}^{n-1};\ \left|\eta' - \xi'\right| < \delta \right\}.$$

Define

$$M_\infty := \sup_{\xi' \in \mathbb{S}^{n-1}, \delta > 0} \left\{ \operatorname*{ess.inf}_{x' \in S_{\xi'}(\delta)} \left(\liminf_{r \to +\infty} \phi(rx') \right) \right\}.$$

Now, we state a main result.

Theorem 1. *Let $n \geq 2$. Assume that $M_\infty > 0$. Then the classical solution for (1) blows up in finite time, and the blow up time is estimated as follows:*

$$T^* \leq \frac{1}{p-1} M_\infty^{1-p}.$$

Once we admit Theorem 1, we can prove the following corollary immediately.

Corollary 1. *Let $n \geq 2$. Suppose that $M_\infty = \|\phi\|_{L^\infty(\mathbb{R}^n)}$. Then the minimal time blow up occurs i.e.,*

$$T^* = \frac{1}{p-1} \|\phi\|_{L^\infty(\mathbb{R}^n)}^{1-p}.$$

In order to prove Theorem 1, we prepare the sequence $\{w_j(t)\}$. For $\xi' \in \mathbb{S}^{n-1}$ and $\delta > 0$, we first determine the sequences $\{a_j\} \subset \mathbb{R}^n$ and $\{R_j\} \subset (0, \infty)$. Let $\{a_j\} \subset \mathbb{R}^n$ be a sequence satisfying that $|a_j| \to \infty$ as $j \to \infty$, and that $a_j/|a_j| = \xi'$ for any $j \in \mathbb{N}$. Put $R_j = (\delta\sqrt{4 - \delta^2}/2)|a_j|$. For $R_j > 0$, let ρ_{R_j} be the first eigenfunction of $-\Delta$ on $B_{R_j}(0) = \{x \in \mathbb{R}^n;\ |x| < R_j\}$ with zero Dirichlet boundary condition under the normalization $\int_{B_{R_j}(0)} \rho_{R_j}(x)dx = 1$. Moreover, let μ_{R_j} be the corresponding first eigenvalue. For the solutions for (1), define

$$w_j(t) := \int_{B_{R_j}(0)} u(x + a_j, t) \rho_{R_j}(x) dx.$$

Now we introduce the following two lemmas.

Lemma 1. *The blow up time of w_j is estimated from above as follows:*

$$T_{w_j}^* \leq \frac{\log\left(1 - \mu_{R_j} w_j^{1-p}(0)\right)}{-(p-1)\mu_{R_j}}$$

for large j.

Lemma 2. (i) *We have*

$$\liminf_{j \to +\infty} w_j(0) \geq \operatorname*{ess.inf}_{x' \in S_{\xi'}(\delta)} \phi_\infty(x').$$

(ii) *We have*

$$\lim_{j \to +\infty} \frac{\log\left(1 - \mu_{R_j} w_j^{1-p}(0)\right)}{-\mu_{R_j} w_j^{1-p}(0)} = 1.$$

From the definition of $w_j(t)$, $T^* \le T^*_{w_j}$ holds for large j. Using the lemmas 1 and 2, we obtain

$$\begin{aligned}
T^* &\le \limsup_{j+\to\infty} T^*_{w_j} \\
&\le \limsup_{j+\to\infty} \frac{\log\left(1-\mu_{R_j} w_j^{1-p}(0)\right)}{-(p-1)\mu_{R_j}} \\
&= \frac{1}{p-1}\lim_{j\to+\infty} \frac{\log\left(1-\mu_{R_j} w_j^{1-p}(0)\right)}{-\mu_{R_j} w_j^{1-p}(0)} \cdot \left(\liminf_{j\to+\infty} w_j(0)\right)^{1-p} \\
&\le \frac{1}{p-1}\left(\operatorname*{ess.inf}_{x'\in S_{\xi'}(\delta)} \phi_\infty(x')\right)^{1-p} \\
&\le \frac{1}{p-1} M_\infty^{1-p}.
\end{aligned}$$

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