Nonlinear diffusion equations associated with $p(x)$-Laplacians

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1 Introduction

Let Ω be a bounded domain of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. This talk is concerned with the Cauchy-Dirichlet problem for nonlinear diffusion equations of the form

$$\partial_t u = \Delta_{p(\cdot)} u + f \quad \text{in } \Omega \times (0,\infty), \tag{1}$$

$$u = 0 \quad \text{on } \partial\Omega \times (0,\infty), \tag{2}$$

$$u(\cdot, 0) = u_0 \quad \text{in } \Omega, \tag{3}$$

where $\partial_t = \partial/\partial t$, $f = f(x,t)$ and $u_0 = u_0(x)$ are given and $\Delta_{p(\cdot)}$ is the so-called $p(\cdot)$-*Laplacian* (or $p(x)$-*Laplacian*) given by

$$\Delta_{p(\cdot)}\phi(x) := \nabla \cdot \left(|\nabla\phi(x)|^{p(x)-2} \nabla\phi(x) \right)$$

with a variable exponent $p(\cdot) : \Omega \to (1,\infty)$.

Notation for variable exponents.

- $p^- := \operatorname*{ess\,inf}_{x\in\Omega} p(x)$ and $p^+ := \operatorname*{ess\,sup}_{x\in\Omega} p(x)$ for variable exponents $p(\cdot) : \Omega \to [1,\infty)$.
- $\mathcal{P}(\Omega) := \left\{ p \in \mathcal{M}(\Omega) \colon 1 \le p^- \le p^+ \le \infty \right\}$.
- $\mathcal{P}^\circ(\Omega) := \left\{ p(\cdot) \in \mathcal{P}(\Omega) \colon 1 < p^- \le p^+ < \infty \right\}$.
- $\mathcal{P}_{\log}(\Omega) := \left\{ p \in \mathcal{P}(\Omega) \colon |p(x) - p(x')| \le \dfrac{L}{\log(|x-x'|^{-1} + e)} \ \forall x, x' \in \Omega \right\}$.

Here $\mathcal{M}(\Omega)$ denotes the set of all (Lebesgue) measurable functions from Ω to $\mathbb{R}$.

2 Variable exponent Lebesgue and Sobolev spaces

In this section, we summarize the definition and several properties of variable exponent Lebesgue and Sobolev spaces, and we refer the reader to a recently published book [4] for a good summary in this field. Variable exponent Lebesgue and Sobolev spaces are defined by

$$L^{p(\cdot)}(\Omega) := \left\{ u \in \mathcal{M}(\Omega) \colon \int_\Omega |u(x)|^{p(x)} dx < \infty \right\},$$

$$W^{1,p(\cdot)}(\Omega) := \left\{ u \in L^{p(\cdot)}(\Omega) \colon \partial_{x_i} u \in L^{p(\cdot)}(\Omega) \ \text{ for } \ i = 1, 2, \ldots, N \right\}$$

with Luxemburg-type norms

$$\|u\|_{p(\cdot)} = \|u\|_{L^{p(\cdot)}(\Omega)} := \inf \left\{ \lambda > 0 \colon \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{p(x)} \mathrm{d}x \le 1 \right\},$$

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} := \left(\|u\|_{p(\cdot)}^2 + \|\nabla u\|_{p(\cdot)}^2 \right)^{1/2}.$$

If $p(\cdot) \in \mathcal{P}^\circ(\Omega)$, then $L^{p(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}(\Omega)$ are uniformly convex. Moreover, it holds that

$$\sigma^-(\|w\|_{p(\cdot)}) \le \int_\Omega |w(x)|^{p(x)} dx \le \sigma^+(\|w\|_{p(\cdot)}) \quad \forall w \in L^{p(\cdot)}(\Omega)$$

with the strictly increasing functions

$$\sigma^-(s) := \min\{s^{p^-}, s^{p^+}\}, \quad \sigma^+(s) := \max\{s^{p^-}, s^{p^+}\} \quad \text{for } s \ge 0.$$

If $p(\cdot) \in \mathcal{P}_{\log}(\Omega)$, then $C_0^\infty(\mathbb{R}^N)$ is dense in $W^{1,p(\cdot)}(\mathbb{R}^N)$, and moreover, Poincaré inequalities and Sobolev embedding theorems hold.

3 Well-posedness

In order to discuss the well-posedness of (1)–(3) without assuming $p(\cdot) \in \mathcal{P}_{\log}(\Omega)$, we introduce the following amalgam spaces:

$$X^{p(\cdot)}(\Omega) := \left\{ u \in L^2(\Omega) \colon \partial u/\partial x_i \in L^{p(\cdot)}(\Omega) \ \text{ for } i = 1, 2, \ldots, N \right\}$$

equipped with the norm

$$\|u\|_{X^{p(\cdot)}(\Omega)} := \left(\|u\|_2^2 + \|\nabla u\|_{p(\cdot)}^2 \right)^{1/2} \qquad \text{for } u \in X^{p(\cdot)}(\Omega).$$

Moreover, set a subspace of $X^{p(\cdot)}(\Omega)$ by

$$X_0^{p(\cdot)}(\Omega) := X^{p(\cdot)}(\Omega) \cap W_0^{1,p^-}(\Omega)$$

with $\|u\|_{X_0^{p(\cdot)}(\Omega)} := \|u\|_{X^{p(\cdot)}(\Omega)}$.

Moreover, we define a functional $\varphi_{p(\cdot)} : L^2(\Omega) \to [0, \infty]$ by

$$\varphi_{p(\cdot)}(w) = \begin{cases} \displaystyle\int_\Omega \frac{1}{p(x)} |\nabla w(x)|^{p(x)} dx & \text{if } \ w \in X_0^{p(\cdot)}(\Omega), \\ \infty & \text{otherwise.} \end{cases} \tag{4}$$

Then we can prove

Lemma 1 (p.l.s.c. of $\varphi_{p_n(\cdot)}$)

Assume $p(\cdot) \in \mathcal{P}^\circ(\Omega)$. Then $\varphi_{p(\cdot)}$ is proper (i.e., $\varphi_{p(\cdot)} \not\equiv \infty$), lower semicontinuous and convex (p.l.s.c. for short) in $L^2(\Omega)$.

Then (1)–(3) is reduced into the Cauchy problem of an evolution equation,

$$\frac{du}{dt}(t) + \partial\varphi_{p(\cdot)}(u(t)) = f(t) \ \text{ in } L^2(\Omega), \quad u(0) = u_0.$$

Here $\partial\varphi_{p(\cdot)}$ denotes the subdifferential of $\varphi_{p(\cdot)}$ and it is defined by

$$\partial\varphi_{p(\cdot)}(u) := \{\xi \in L^2(\Omega) \colon \varphi_{p_n(\cdot)}(v) - \phi_{p_n(\cdot)}(u) \geq (\xi, v - u)_{L^2} \ \forall v \in L^2(\Omega)\}.$$

From a general theory, we have

Theorem 2 (Well-posedness)

Let $p(\cdot) \in \mathcal{P}^\circ(\Omega)$, $f \in L^2_{loc}([0, \infty); L^2(\Omega))$ and assume that $1 < p^-$ and $p^+ < \infty$.

- For $u_0 \in L^2(\Omega)$, the Cauchy-Dirichlet problem (1)–(3) admits a unique solution $u = u(x, t) \in C((0, \infty); X_0^{p(\cdot)}(\Omega))$.
- If $u_0 \in X_0^{p(\cdot)}(\Omega)$, then $u \in W^{1,2}(0, T; L^2(\Omega)) \cap C([0, T]; X_0^{p(\cdot)}(\Omega))$ for any $T > 0$.
- The unique solution u continuously depends on initial data u_0,

  $$\|u_1(t) - u_2(t)\|_2 \leq \|u_{0,1} - u_{0,2}\|_2 \qquad \text{for all } \ t \geq 0,$$

  where u_i is a unique solution of (1)–(3) for the initial data $u_{0,i}$ $(i = 1, 2)$.

4 Fast/slow diffusion limit

Let $p_n(\cdot)$ be a sequence in $\mathcal{P}^\circ(\Omega)$ such that

$$p_n(x) \to \infty \qquad \text{for a.e. } x \in \Omega.$$

In this section, we discuss the limiting behavior as $n \to \infty$ of the solutions $u_n = u_n(x,t)$ for

$$\partial_t u_n = \Delta_{p_n(\cdot)} u_n + f \quad \text{in } \Omega \times (0,T), \tag{5}$$

$$u_n = 0 \quad \text{on } \partial\Omega \times (0,T), \tag{6}$$

$$u_n(\cdot,0) = u_{0,n} \quad \text{in } \Omega \tag{7}$$

with a constant $T > 0$ and initial data $u_{0,n} \to u_0$ in a proper sense. Here we pay attention to the Mosco convergence as $n \to \infty$ of $\varphi_{p_n(\cdot)}$, which is given as in (4), under an appropriate assumption and a limit functional

$$\varphi_\infty(w) = I_K(w) = \begin{cases} 0 & \text{if } w \in K, \\ \infty & \text{else} \end{cases}$$

with $K := \left\{ w \in H^1_0(\Omega) \colon \|\nabla w\|_\infty \le 1 \right\}$. Then the limiting problem is written as the following variational inequality:

$$\begin{aligned} \int_\Omega (f(x,t) - \partial_t u(x,t)) \big(v(x) - u(x,t)\big) dx \le 0 \quad & \text{for a.e. } t \in (0,T) \text{ and all } v \in K, \\ u(\cdot,t) \in K \quad & \text{for a.e. } t \in (0,T), \\ u(\cdot,0) = u_0 \quad & \text{in } \Omega. \end{aligned}$$

More precisely, we obtain

Proposition 3 (Mosco convergence of $\varphi_{p_n(\cdot)}$)

Assume that

$$p_n^- \to \infty \quad \text{and} \quad (p_n^+)^{1/p_n^-} \to 1 \quad \text{as } n \to \infty.$$

Then $\varphi_{p_n(\cdot)} \to \varphi_\infty$ on $L^2(\Omega)$ in the sense of Mosco as $n \to \infty$.

By using a general theory of structural stability of evolution equations with Mosco-convergent functionals (see [3]), one can prove

Theorem 4 (Convergence of solutions)

In addition to the same assumption, assume that

$$u_{0,n} \to u_0 \in K \ \text{ strongly in } L^2(\Omega) \ \text{ and } \ \varphi_{p_n(\cdot)}(u_{0,n}) \to 0.$$

Then solutions u_n of (5)–(7) converge strongly in $W^{1,2}(0,T;L^2(\Omega))$ to a solution of the variational inequality above.

We finally give a definition of Mosco convergence. Let H be a Hilbert space and denote *effective domain* by

$$D(\phi) := \{u \in H \colon \phi(u) < \infty\}$$

for $\phi : H \to (-\infty, \infty]$ proper, lower semicontinuous and convex (p.l.s.c.).

Definition 5 (Mosco convergence)

Let ϕ_n and ϕ be p.l.s.c. on H. We say $\phi_n \to \phi$ on H in the sense of Mosco as $n \to \infty$, if the following conditions are all satisfied:

(i) For all $u \in D(\phi)$, there exists $u_n \in D(\phi_n)$ such that $u_n \to u$ strongly in H and $\phi_n(u_n) \to \phi(u)$.

(ii) If $u_n \to u$ weakly in H, then $\liminf_{n\to\infty} \phi_n(u_n) \ge \phi(u)$.

5 Partial fast/slow diffusion limit

We next treat partially divergent exponents of the form,

$$p_n(x) = \begin{cases} q_n(x) \to \infty & \text{if } x \in D, \\ q(x) < \infty & \text{if } x \in \Omega \setminus \overline{D} \end{cases} \qquad \text{as } n \to \infty,$$

where D is a non-empty open subset of Ω satisfying $\Omega \setminus \overline{D} \neq \emptyset$, $q_n \in \mathcal{P}^\circ(D)$ and $q \in \mathcal{P}^\circ(\Omega \setminus \overline{D})$. Here we introduce a functional $\varphi_D : L^2(\Omega) \to [0, \infty]$ given by

$$\varphi_D(w) := \begin{cases} \displaystyle\int_{\Omega\setminus\overline{D}} \frac{1}{q(x)} |\nabla w(x)|^{q(x)} dx & \text{if } w \in W_0^{1,q^-}(\Omega), \quad w \in X^{q(\cdot)}(\Omega \setminus \overline{D}), \\ & \quad \text{and } \|\nabla w\|_{L^\infty(D)} \leq 1, \\ \infty & \text{otherwise} \end{cases}$$

as a limit of $\varphi_{p_n(\cdot)}$. Our result is state in the following, where we write

$$q_n^+ := \operatorname*{ess\,sup}_{x \in D} q_n(x) \quad \text{and} \quad q_n^- := \operatorname*{ess\,inf}_{x \in D} q_n(x).$$

Proposition 6 (Mosco convergence of $\varphi_{p_n(\cdot)}$)

Assume that

$$q_n^- \to \infty \quad \text{and} \quad (q_n^+)^{1/q_n^-} \to 1 \quad \text{as } n \to \infty.$$

Then $\varphi_{p_n(\cdot)} \to \varphi_D$ on $L^2(\Omega)$ in the sense of Mosco as $n \to \infty$.

From the Mosco convergence of $\varphi_{p_n(\cdot)}$, solutions u_n converge to u strongly in $W^{1,2}(0, T; L^2(\Omega))$ and the limit u solves

$$\frac{du}{dt}(t) + \partial\varphi_D(u(t)) \ni f(t) \text{ in } L^2(\Omega), \quad u(0) = u_0.$$

We further characterize the limiting problem above as follows:

- Properties of $u(t)$ at each t:

$$u(t) \in W_0^{1,q^-}(\Omega), \quad u(t) \in X^{q(\cdot)}(\Omega \setminus \overline{D}), \quad \|\nabla u(t)\|_{L^\infty(D)} \leq 1 \quad \text{for a.e. } t \in (0, T).$$

- A parabolic equation in $\Omega \setminus \overline{D}$:

$$\partial_t u - \Delta_{q(\cdot)} u = f \quad \text{in } \mathscr{D}'(\Omega \setminus \overline{D}) \text{ and } t > 0.$$

- A variational inequality in D:

$$\int_D (f(x,t) - \partial_t u(x,t))(z(x) - u(x,t)) dx \leq 0 \quad \forall z \in K_D(u(t)),$$

  where the set $K_D(w)$ is given for each $w \in W_0^{1,q^-}(\Omega)$ by

$$K_D(w) := \left\{ z \in W^{1,\infty}(D) \colon z - w \in W_0^{1,q^-}(D) \text{ and } \|\nabla z\|_{L^\infty(D)} \leq 1 \right\}.$$

References

[1] Akagi, G. and Matsuura, K., Nonlinear diffusion equations driven by the $p(\cdot)$-Laplacian, accepted.

[2] Akagi, G. and Matsuura, K., Well-posedness and large-time behaviors of solutions for a parabolic equation involving $p(x)$-Laplacian, a suppl. volume of Disc. Conti. Dyn. Sys, 2011, pp.22–31.

[3] Attouch, H., *Variational Convergence for Functions and Operators*, Applicable Mathematics Series, Pitman (Advanced Publishing Program), Boston, MA, 1984.

[4] Diening, L., Harjulehto, P., Hästö, P. and Růžička, M., *Lebesgue and Sobolev spaces with variable exponents*, Lecture Notes in Mathematics, vol. 2017, Springer-Verlag, Berlin, 2011.