Variational Inequalities for a System of Elliptic-Parabolic Equations

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§1 Introduction

This is a joint work with Masahiro KUBO[2] and Ken SHIRAKAWA[3].

In this talk, we consider the following vector-valued elliptic-parabolic variational inequality with time-dependent constraint, denoted by $(\mathrm{P})_m$ $(m \geq 1)$:

Problem $(\mathrm{P})_m$.

$$\boldsymbol{u}(t) \in \boldsymbol{K}(t), \qquad 0 < t < T,$$

$$\left(\frac{d}{dt}\boldsymbol{b}(\boldsymbol{u}(t)),\ \boldsymbol{u}(t) - \boldsymbol{z}\right)_{\boldsymbol{H}} + \int_\Omega \boldsymbol{a}(x, \boldsymbol{b}(\boldsymbol{u}(t)), \nabla \boldsymbol{u}(t)) \cdot \nabla(\boldsymbol{u}(t) - \boldsymbol{z})\, dx \leq (\boldsymbol{f}(t),\ \boldsymbol{u}(t) - \boldsymbol{z})_{\boldsymbol{H}}$$

$$\text{for all } \boldsymbol{z} \in \boldsymbol{K}(t) \text{ and } t \in (0, T),$$

$$\boldsymbol{b}(\boldsymbol{u}(0)) = \boldsymbol{b}_0 \quad \text{in } \Omega.$$

Here, $m \geq 1$ is a positive integer, $\boldsymbol{u} = (u_1, \cdots, u_m)$ and $\boldsymbol{b}(\boldsymbol{u}) = (b_1(\boldsymbol{u}), \cdots, b_m(\boldsymbol{u}))$. Also, T is a fixed finite time, Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 1)$ with a smooth boundary, $(\cdot, \cdot)_{\boldsymbol{H}}$ is the usual inner product in $\boldsymbol{H} := [L^2(\Omega)]^m$, the constraint $\boldsymbol{K}(t)$ is a time-dependent convex set in $\boldsymbol{V} := [H^1(\Omega)]^m$, $\boldsymbol{f}$ is a given function in $W^{1,2}(0, T; \boldsymbol{H})$, and $\boldsymbol{b}_0 \in \boldsymbol{H}$ is a given initial value. For the quasilinear elliptic vector field $\boldsymbol{a}(x, s, p)$ and the nonlinear term $\boldsymbol{b}$, we assume structural conditions. In particular, we assume $\boldsymbol{a}(x, s, p) = \partial_p A(x, s, p)$ and $\boldsymbol{b} = \partial B$ for potential functions $A : \Omega \times \mathbb{R}^m \times \left(\mathbb{R}^N\right)^m \to \mathbb{R}$ and $B : \mathbb{R}^m \to \mathbb{R}$, respectively.

In the case when $\boldsymbol{a}(x, s, p)$ is strictly elliptic in p, the system $(\mathrm{P})_m$ was studied by Alt–Luckhaus [2] for the Dirichlet-Neumann boundary condition, namely, the constraint $\boldsymbol{K}(t)$ is given by

$$\boldsymbol{K}(t) = \{\boldsymbol{z} \in \boldsymbol{V}\ ;\ \boldsymbol{z} = \boldsymbol{p}(t) \text{ on } \Gamma_D\},$$

where Γ_D is a part of the boundary of Ω and $\boldsymbol{p}$ is given boundary data on Γ_D.

For $m = 1$, there is already a vast literature (cf. Kubo-Yamazaki [6] and the references therein). However, the case $m > 1$ seems to have been studied less extensively so far.

The main aim of this talk is to study the system $(\mathrm{P})_m$ $(m \geq 1)$ with a general time-dependent convex constraint imposed by $\boldsymbol{K}(t)$. In fact, we establish a solvability theorem concerning the existence of solutions to $(\mathrm{P})_m$ by employing an improved version of the method from Kubo–Yamazaki [6].

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§2 Main Theorem

Now, we assume the following conditions.

(A1) $A : \Omega \times \mathbb{R}^m \times \left(\mathbb{R}^N\right)^m \to \mathbb{R}$ is a $\mathcal{C}^1$-class function such that $\boldsymbol{a}(x, s, p) = \partial_p A(x, s, p)$,

$$\boldsymbol{a}(x, \cdot, \cdot) : \mathbb{R}^m \times \left(\mathbb{R}^N\right)^m \to \left(\mathbb{R}^N\right)^m \text{ is continuous for a.e. } x \in \Omega,$$

$$\boldsymbol{a}(\cdot, s, p) : \Omega \to \left(\mathbb{R}^N\right)^m \text{ is measurable for any } s \in \mathbb{R}^m,\ p \in \left(\mathbb{R}^N\right)^m,$$

$$A(x, \cdot, \cdot) : \mathbb{R}^m \times \left(\mathbb{R}^N\right)^m \to \mathbb{R} \text{ is continuous for a.e. } x \in \Omega,$$

$$A(\cdot, s, p) : \Omega \to \mathbb{R} \text{ is measurable for any } s \in \mathbb{R}^m,\ p \in \left(\mathbb{R}^N\right)^m,$$

$$A(x, s, \cdot) : \left(\mathbb{R}^N\right)^m \to \mathbb{R} \text{ is convex for any } x \in \Omega, s \in \mathbb{R}^m.$$

Moreover, there exist constants $C_1 > 0$ and $C_2 > 0$ such that

$$|\boldsymbol{a}(x, s, p)|^2 + |A(x, s, p)| + |\partial_s A(x, s, p)|^2 \le C_1(1 + |s|^2 + |p|^2),$$

$$A(x, s, p) \ge C_2|p|^2$$

for all $x \in \Omega,\ s \in \mathbb{R}^m$ and $p \in \left(\mathbb{R}^N\right)^m$.

(A2) $B : \mathbb{R}^m \to \mathbb{R}$ is a $\mathcal{C}^1$-class convex function such that $\boldsymbol{b} = \partial B$ is Lipschitz continuous.

(A3) $\boldsymbol{f} \in W^{1,2}(0, T; \boldsymbol{H})$ and $\boldsymbol{K}(t)$ is a non-empty, closed, convex set in $\boldsymbol{V}$ for all $t \in [0, T]$. Also, $\boldsymbol{b}_0 = \boldsymbol{b}(\boldsymbol{u}_0)$ for some $\boldsymbol{u}_0 \in \boldsymbol{K}(0)$.

(A4) There is a function $\alpha \in W^{1,2}(0, T)$ satisfying the following property: for any $s, t \in [0, T]$, $\boldsymbol{w} \in \boldsymbol{H}$ and $\boldsymbol{z} \in \boldsymbol{K}(s)$, there exists $\widetilde{\boldsymbol{z}} \in \boldsymbol{K}(t)$ such that

$$|\widetilde{\boldsymbol{z}} - \boldsymbol{z}|_{\boldsymbol{V}} \le |\alpha(t) - \alpha(s)|(1 + |\boldsymbol{z}|_{\boldsymbol{V}}),$$

$$\int_\Omega A(x, \boldsymbol{w}, \nabla\widetilde{\boldsymbol{z}})\, dx - \int_\Omega A(x, \boldsymbol{w}, \nabla\boldsymbol{z})\, dx \le |\alpha(t) - \alpha(s)|(1 + |\boldsymbol{z}|_{\boldsymbol{V}}^2 + |\boldsymbol{w}|_{\boldsymbol{H}}|\boldsymbol{z}|_{\boldsymbol{V}} + |\boldsymbol{w}|_{\boldsymbol{H}}).$$

(A5) There is a constant $C_3 > 0$ such that

$$|\boldsymbol{z}|_{\boldsymbol{V}} \le C_3(1 + |\nabla\boldsymbol{z}|_{\boldsymbol{H}}) \quad \text{for all } \boldsymbol{z} \in \boldsymbol{K}(t) \text{ and } t \in [0, T].$$

We now mention the main theorem concerning the existence of solutions to $(\mathrm{P})_m$ $(m \ge 1)$.

Main Theorem. Assume (A1)–(A5) are satisfied. Then, there is at least one solution $\boldsymbol{u} : [0, T] \to \boldsymbol{V}$ to $(\mathrm{P})_m$ such that $\boldsymbol{u} \in L^\infty(0, T; \boldsymbol{V})$ and $\boldsymbol{b}(\boldsymbol{u}) \in W^{1,2}(0, T; \boldsymbol{H})$.

§3 Application to a regularization system of oil and water problem

In this section we give an application of Main Theorem.

Now, we consider the following regularization system of oil and water problem:

Problem (P1).

$$s_i(u_1-u_2)_t - \nabla\cdot(\nabla u_i + k_i(s_1(u_1-u_2))\boldsymbol{e}_i) = f_i(t,x) \qquad \text{in } (0,T)\times\Omega,$$

$$s_1+s_2=1 \qquad \text{in } (0,T)\times\Omega,$$

$$u_i \le p_i, \qquad \nu\cdot(\nabla u_i + k_i(s_1(u_1-u_2))\boldsymbol{e}_i) \le 0$$

$$\text{and} \qquad (u_i-p_i)\nu\cdot(\nabla u_i + k_i(s_1(u_1-u_2))\boldsymbol{e}_i) = 0 \qquad \text{on } (0,T)\times\Gamma_{i,S},$$

$$u_i = p_i \qquad \text{on } (0,T)\times\Gamma_{i,D},$$

$$\nu\cdot(\nabla u_i + k_i(s_1(u_1-u_2))\boldsymbol{e}_i) = 0 \qquad \text{on } (0,T)\times\Gamma_{i,N},$$

$$s_i(u_1(0)-u_2(0)) = b_{i,0} \quad \text{in } \Omega$$

for $i=1,2$, where Ω is a bounded domain in $\mathbb{R}^N$ $(N\ge 1)$ having Lipschitz boundary when $N>1$, $\boldsymbol{e}_i$ is a vector in x_N-direction, ν is the outward normal vector on the boundary, and the functions p_i, $b_{i,0}$ are given $(i=1,2)$. Also, the boundary Γ of Ω admits the mutually disjoint decomposition

$$\Gamma = \Gamma_{i,S}\cup\Gamma_{i,D}\cup\Gamma_{i,N}, \quad (i=1,2),$$

where $\Gamma_{i,S},\Gamma_{i,D}$ and $\Gamma_{i,N}$ are $\mathscr{H}^{N-1}$-measurable subsets of Γ, and $\Gamma_{i,D}$ has positive $\mathscr{H}^{N-1}$-measure $(i=1,2)$.

In physical applications, Ω is the porous medium, the indices 1 and 2 relate to the single fluids: water and oil. Also, s_i stands for the saturation, u_i is the hydrostatic pressure, and k_i is the hydraulic conductivity (cf. [1, 3, 4]).

Here, we assume that

(K1) $\boldsymbol{p}=(p_1,p_2)\in W^{1,2}(0,T;\boldsymbol{V})$ and $\boldsymbol{f}=(f_1,f_2)\in W^{1,2}(0,T;\boldsymbol{H})$.

(K2) $s_1:\mathbb{R}\to\mathbb{R}$ is a nondecreasing and Lipschitz continuous function.

(K3) $k_i:\mathbb{R}\to\mathbb{R}$ is a bounded and Lipschitz continuous function $(i=1,2)$.

We easily see that (P1) can be reformulated to Problem $(\mathrm{P})_2$. In fact, for each $t\in[0,T]$ we define a convex set $\boldsymbol{K}_1(t)$ in $\boldsymbol{V}$ by

$$\boldsymbol{K}_1(t) := \left\{ \boldsymbol{z}=(z_1,z_2)\in\boldsymbol{V} \;\middle|\; \begin{array}{c} z_i\le p_i(t) \text{ on } \Gamma_{i,S} \text{ and } z_i=p_i(t) \text{ on } \Gamma_{i,D} \\ \text{for } i=1,2 \end{array} \right\}.$$

Here, we put $\boldsymbol{u}:=(u_1,u_2)$ and $\boldsymbol{b}(\boldsymbol{u}):=(s_1(u_1-u_2),1-s_1(u_1-u_2))$ in $\boldsymbol{H}$. Clearly, we have

$$\boldsymbol{b}(\boldsymbol{u})' = (s_1(u_1-u_2)',(1-s_1(u_1-u_2))') = (s_1(u_1-u_2)',s_2(u_1-u_2)') \quad \text{in } \boldsymbol{H}.$$

Also, we define

$$\boldsymbol{a}(x,\boldsymbol{w},\nabla\boldsymbol{u}(t)) := (\nabla u_1 + k_1(w_1)\boldsymbol{e}_1, \nabla u_2 + k_2(w_1)\boldsymbol{e}_2) \quad \text{for } \boldsymbol{w}=(w_1,w_2)\in\mathbb{R}^2.$$

Then, we easily observe that Problem $(\mathrm{P})_2$ with $\boldsymbol{K}(t) = \boldsymbol{K}_1(t)$ is the weak variational formulation of (P1).

Now, we show (A1). To do so, we define

$$A(x, \boldsymbol{w}, \boldsymbol{v}) := \sum_{i=1}^{2} \left[\frac{1}{2}\boldsymbol{v}_i + k_i(w_1)\boldsymbol{e}_i\right] \cdot \boldsymbol{v}_i + C_A$$

for all $x \in \Omega$, $\boldsymbol{w} = (w_1, w_2) \in \mathbb{R}^2$ and $\boldsymbol{v} = (\boldsymbol{v}_1, \boldsymbol{v}_2) \in \mathbb{R}^N \times \mathbb{R}^N$, where C_A is a positive constant so that

$$A(x, \boldsymbol{w}, \boldsymbol{v}) \geq \frac{1}{4}|\boldsymbol{v}|^2.$$

Then, we easily observe from (K3) that the assumption (A1) holds.

Next, we show (A2). Now, we define

$$B(\boldsymbol{u}) = \int_0^{u_1 - u_2} s_1(\rho) d\rho + u_2.$$

Then, we easily observe from (K2) that $B : \mathbb{R}^2 \to \mathbb{R}$ is a C^1-class convex function satisfying $\boldsymbol{b} = \partial B$. Therefore, the assumption (A2) holds.

Also, the assumption (A4) is verified by (K1), (K3) and by putting $\widetilde{\boldsymbol{z}} := \boldsymbol{z} - \boldsymbol{p}(s) + \boldsymbol{p}(t)$ for $\boldsymbol{z} \in \boldsymbol{K}_1(s)$. Condition (A5) is easily checked by noting that $\Gamma_{i,D}$ has positive $\mathscr{H}^{N-1}$-measure ($i = 1, 2$) and by using the Poincaré inequality. Hence, if $\boldsymbol{u}_0 = (u_{1,0}, u_{2,0}) \in \boldsymbol{K}_1(0)$ and $\boldsymbol{b}_0 = (b_{1,0}, b_{2,0}) = (s_1(u_{1,0} - u_{2,0}), 1 - s_1(u_{1,0} - u_{2,0}))$, we can apply Main Theorem to Problem (P1). Thus, we get the existence of solutions to (P1).

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