LOGARITHMIC TIME DECAY FOR CUBIC NONLINEAR SCHRÖDINGER EQUATIONS

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1. Introduction

We study the Cauchy problem for the one dimensional cubic nonlinear Schrödinger equation

$$\left\{\begin{array}{c} iu_t + \frac{1}{2}u_{xx} = \mathcal{N},\ x \in \mathbf{R},\ t > 1, \\ u(1,x) = u_0(x),\ x \in \mathbf{R}, \end{array}\right. \tag{1.1}$$

where $\mathcal{N} = \lambda e^{i\frac{\pi}{2}} u^3 + |u|^2 u,\ 0 < |\lambda| < \sqrt{3}$. A more general cubic nonlinear Schrödinger equation

$$\left\{\begin{array}{c} iu_t + \frac{1}{2}u_{xx} = \lambda e^{i\omega} u^3 + \kappa |u|^2 u,\ x \in \mathbf{R},\ t > 1, \\ u(1,x) = u_0(x),\ x \in \mathbf{R}, \end{array}\right.$$

where $\lambda, \omega, \kappa \in \mathbf{R}$, $\lambda, \kappa \neq 0$, can be reduced to (1.1) by changing the dependent variable $u = \frac{1}{\kappa} e^{i\frac{\gamma}{2} - i\frac{\pi}{4}} v$, and replacing $\frac{\lambda}{\kappa}$ by λ.

2. Main result

Our main result is the following. Denote $v_0(\xi) = ie^{-\frac{i}{2}\xi^2} u_0(\xi)$, $w_0(\xi) = v_0'(\xi) + \phi(\xi) v_0^3(\xi)$, and

$$\phi(\xi) = \frac{2\lambda}{\sqrt{3}} e^{-\frac{i}{2}\xi^2} \int_0^\xi \left(e^{\frac{i}{2}\eta^2} - \sqrt{3} e^{3\frac{i}{2}\eta^2} \right) d\eta.$$

Theorem 2.1. *Let $0 < |\lambda| < \sqrt{3}$. Let the initial data $v_0 \in \mathbf{H}^2$ and the norm $\|v_0\|_{\mathbf{L}^\infty} \le \epsilon$ and $\|w_0\|_{\mathbf{H}^1} \le \epsilon^4$, where $\epsilon > 0$ is sufficiently small. Also suppose that*

$$|v_0(0)| \ge \delta, \tag{2.1}$$

where $\delta = \epsilon^{1+\nu}$ with small $\nu > 0$. Then there exists a unique solution $u \in \mathbf{C}\left([1,\infty); \mathbf{L}^2\right)$ of the Cauchy problem (1.1). Moreover the time decay estimates

$$\begin{aligned} C_1 \delta t^{-\frac{1}{2}} \left(1 + \epsilon^4 \log t\right)^{-\frac{1}{4}} &\le \sup_{|x| \le \sqrt{t}} |u(t,x)| \\ &\le C_2 \epsilon t^{-\frac{1}{2}} \left(1 + \delta^4 \log t\right)^{-\frac{1}{4}} \end{aligned}$$

and

$$\sup_{|x| > \sqrt{t}} |u(t,x)| \le C\epsilon t^{-\frac{1}{2}}$$

are valid for large t.

We do not know the last estimate is sharp or not.

Key words and phrases. This is a joint work with P.I.Naumkin.

Remark 2.1. *For example, we can choose the initial data as follows* $u_0(\xi) = -ie^{\frac{i}{2}\xi^2} v_0(\xi)$ *and*

$$v_0(\xi) = \frac{\delta}{\sqrt{1 + 2\delta^2 \int_0^\xi \phi(\eta)\, d\eta + \epsilon^{12}\xi^2}}.$$

Then

$$w_0(\xi) = -\frac{\epsilon^{12}\xi}{\delta^2} v_0^3(\xi)$$

satisfies the estimate

$$\|w_0\|_{\mathbf{H}^1} \leq C\epsilon^6\delta \left\| \left(1 + \epsilon^{12}\xi^2\right)^{-1} \right\|_{\mathbf{L}^2} \leq C\epsilon^3\delta \leq \epsilon^4.$$

Remark 2.2. *Condition* $|v_0(0)| \geq \delta$ *of Theorem 2.1 excludes the vanishing condition at the origin for the final data* $\widehat{u}_+(0) = 0$, *which was assumed in papers* [1], [2], [3], [4], [5]. *The condition* $0 < |\lambda| < \sqrt{3}$ *is essential, since we believe that the asymptotic behavior is different for the case of* $|\lambda| \geq \sqrt{3}$.

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