Well-posedness for a system of quadratic derivative nonlinear Schrödinger equations at the scaling critical regularity

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We consider the Cauchy problem of the system of Schrödinger equations:

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\text{(SS)}\begin{cases}(i\partial_t+\alpha\Delta)u=-(\nabla\cdot w)v, & t>0,\ x\in\mathbb{R}^d\\ (i\partial_t+\beta\Delta)v=-(\nabla\cdot\overline{w})u, & t>0,\ x\in\mathbb{R}^d\\ (i\partial_t+\gamma\Delta)w=\nabla(u\cdot\overline{v}), & t>0,\ x\in\mathbb{R}^d\\ (u,v,w)|_{t=0}=(u_0,v_0,w_0)\in H^s(\mathbb{R}^d)\times H^s(\mathbb{R}^d)\times H^s(\mathbb{R}^d)\end{cases}.
$$

Where α, β, $\gamma\neq 0$ and unknown functions u, v, w are $\mathbb{C}^d$ value. System (SS) is introduced by M.Colin and T.Colin([1]) as a model of laser-plasma interaction. M.Colin and T.Colin([1]) also proved the local existence of (SS) for $s>d/2+3$. Our purpose is to improve their result and to prove the well-posedness of (SS) in the scaling critical Sobolev space.

System (SS) is invariant under the following scaling transformation:

$$A_\lambda(t,x)=\lambda^{-1}A(\lambda^{-2}t,\lambda^{-1}x)\ \ (A=(u,v,w)).$$

We note that

$$||A_\lambda(0,\cdot)||_{\dot{H}^s}=\lambda^{d/2-1-s}||A(0,\cdot)||_{\dot{H}^s}$$

and the scaling critical regularity of (SS) is $s=s_c:=d/2-1$.

We put $\phi:=(\alpha-\gamma)(\beta+\gamma)$ and $\theta:=\alpha\beta\gamma(1/\alpha-1/\beta-1/\gamma)$. Main results in this talk are the following.

Theorem 1 (For the critical result)**.** *We assume $\phi\neq 0$.*

(1) *If $d\geq 4$, then* (SS) *is globally well-posed for small data in H^{s_c}.*
(2) *If $d=2$, 3 and $\theta>0$, then* (SS) *is globally well-posed for small data in H^{s_c}.*

Theorem 2 (For the subcritical result)**.** *We assume $\phi\neq 0$.*

(1) *If $d=1$ and $\theta>0$, then* (SS) *is locally well-posed in L^2.*
(2) *If $d=1$, 2, 3 and $\theta=0$, then* (SS) *is locally well-posed in H^1.*
(3) *If $d=2$, 3 and $\theta<0$, then* (SS) *is locally well-posed in $H^{1/2+\epsilon}$ for any $\epsilon>0$.*
(4) *If $d=1$ and $\theta<0$, then* (SS) *is locally well-posed in $H^{1/2}$.*

Theorem 3 (Negative result)**.** *Let $d\geq 1$.*

(1) *If $\phi=0$, then flow map of* (SS) *is not C^2 in H^s for any $s\in\mathbb{R}$.*
(2) *If $\theta=0$, then flow map of* (SS) *is not C^2 in H^s for any $s<1$.*
(3) *If $\theta<0$, then flow map of* (SS) *is not C^2 in H^s for any $s<1/2$.*

Remark 4. *System* (SS) *has the following conservation quantities:*

$$M(u,v,w):=2||u||_{L^2_x}^2+||v||_{L^2_x}^2+||w||_{L^2_x}^2,$$
$$H(u,v,w):=\alpha||\nabla u||_{L^2_x}^2+\beta||\nabla v||_{L^2_x}^2+\gamma||\nabla w||_{L^2_x}^2+2\mathrm{Re}(w,\nabla(u\cdot\overline{v}))_{L^2_x}.$$

By using the conservation law for M, we can extend the local L^2 solution of Theorem 2 *globally in time. Furthermore, if α, β and γ are same sign, then we can extend the local H^1 solution of* Theorem 2 *globally in time by using the conservation law for H.*

		$d=1$	$d=2,3$	$d\geq 4$
$\phi\neq 0$	$\theta>0$	WP for $s\geq 0$	WP for $s\geq s_c$	WP for $s\geq s_c$
	$\theta=0$	WP for $s\geq 1$ & not C^2 for $s<1$		
	$\theta<0$	WP for $s\geq 1/2$ & not C^2 for $s<1/2$	WP for $s>1/2$ & not C^2 for $s<1/2$	
$\phi=0$		not C^2 for any $s\in\mathbb{R}$		

TABLE 1. Well-posedness (WP for short) for above the scaling critical regularity

The difficulty is that there is a derivative loss arising from the nonlinear terms. To recover the derivative loss completely, we use the U^2, V^2 type Bourgain spaces which are applied to prove the well-posedness of KP-II equation in the scaling critical Sobolev space by M.Hadac, S.Herr and H.Koch([2]).

To introduce the U^p space and V^p space, we define the set of finite partitions $\mathcal{Z}$ as

$$\mathcal{Z} := \left\{\{t_k\}_{k=0}^K | K\in\mathbb{N}, -\infty=t_0<t_1<\cdots<t_K=\infty\right\}.$$

Definition 5 (U^p space)**.** *Let $1\leq p<\infty$. For $\{t_k\}_{k=0}^K\in\mathcal{Z}$ and $\{\phi_k\}_{k=0}^{K-1}\subset L^2$ with $\sum_{k=0}^{K-1}||\phi_k||_{L^2}^p=1$ and $\phi_0=0$ we call the function $a:\mathbb{R}\to L^2$ given by*

$$a(t)=\sum_{k=1}^K \mathbf{1}_{[t_{k-1},t_k)}(t)\phi_{k-1}$$

a "U^p-atom". Furthermore, we define the atomic space

$$U^p := \left\{ u=\sum_{j=1}^\infty \lambda_j a_j \text{ in } L^\infty(\mathbb{R};L^2) \,\middle|\, a_j: U^p\text{−atom},\ \lambda_j\in\mathbb{C} \text{ such that } \sum_{j=1}^\infty|\lambda_j|<\infty \right\}$$

with norm

$$||u||_{U^p} := \inf\left\{ \sum_{j=1}^\infty |\lambda_j| \,\middle|\, u=\sum_{j=1}^\infty\lambda_j a_j,\ a_j: U^p\text{−atom},\ \lambda_j\in\mathbb{C}\right\}.$$

Definition 6 (V^p space)**.** *Let $1\leq p<\infty$. We define the space of bounded p-variation*

$$V^p := \{v:\mathbb{R}\to L^2 |\ \lim_{t\to-\infty} v(t) \text{ and } \lim_{t\to\infty} v(t) \text{ exist},\ ||v||_{V^p}<\infty\}$$

with norm

$$||v||_{V^p} := \sup_{\{t_k\}_{k=0}^K\in\mathcal{Z}} \left(\sum_{k=1}^K ||v(t_k)-v(t_{k-1})||_{L^2}^p\right)^{1/p},$$

where $v(-\infty):=\lim_{t\to-\infty}v(t)$ and $v(\infty):=0$. Likewise, let $V_{-,rc}^p$ denote the closed subspace of all right-continuous functions $v\in V^p$ with $\lim_{t\to-\infty}v(t)=0$.

Definition 7 (U^2, V^2 type Bourgain spaces)**.** *Let $s,\sigma\in\mathbb{R}$. We define the function space Z_σ^s as the closure of all $u\in C(\mathbb{R};H^s(\mathbb{R}^d))\cap U_\sigma^2$ such that*

$$||u||_{Z_\sigma^s} := \left(\sum_{N\geq 1} N^{2s}||P_N u||_{U_\sigma^2}^2\right)^{1/2} < \infty$$

with respect to the $||\cdot||_{Z_\sigma^s}$-norm. Where

$$U_\sigma^p := \{u:\mathbb{R}\to L^2 |\ e^{-it\sigma\Delta}u\in U^p\},\ ||u||_{U_\sigma^p} := ||e^{-it\sigma\Delta}u||_{U^p}$$

and P_N is (inhomogeneous) Littlewood-Paley decomposition operator with x. We also define by Y_σ^s corresponding space where U^2 is replaced by $V_{-,rc}^2$.

Key estimates to prove the critical result are the following.

Proposition 8 (Key estimates)**.** *We define*

$$I^{(1)}_{T,\sigma}(f,g)(t) := \int_0^t \mathbf{1}_{[0,T)}(t')e^{i(t-t')\sigma\Delta}(\nabla\cdot f(t'))g(t')dt',$$
$$I^{(2)}_{T,\sigma}(f,g)(t) := \int_0^t \mathbf{1}_{[0,T)}(t')e^{i(t-t')\sigma\Delta}\nabla(f(t')\cdot g(t'))dt'$$

and assume $(\alpha-\gamma)(\beta+\gamma)\neq 0$

(1) *If* $d\geq 4$*, then for any* $0<T<\infty$ *we have*

$$||I^{(1)}_{T,\alpha}(w,v)||_{Z^{s_c}_\alpha} \lesssim ||w||_{Y^{s_c}_\gamma}||v||_{Y^{s_c}_\beta},$$
$$||I^{(1)}_{T,\beta}(\overline{w},u)||_{Z^{s_c}_\beta} \lesssim ||w||_{Y^{s_c}_\gamma}||u||_{Y^{s_c}_\alpha},$$
$$||I^{(2)}_{T,\gamma}(u,\overline{v})||_{Z^{s_c}_\gamma} \lesssim ||u||_{Y^{s_c}_\alpha}||v||_{Y^{s_c}_\beta}.$$

(2) *If* $d=2,\,3$ *and* $\alpha\beta\gamma(1/\alpha-1/\beta-1/\gamma)>0$*, then for any* $0<T\leq 1$ *above estimates hold.*

We will talk about the properties of U^p, V^p and the outline of the proof of Proposition 8.

References

[1] M. Colin and T. Colin, *On a quasilinear Zakharov sysrem describing laser-plasma interactions,* Differential Integral Equations. **17**(2004), 297–330.

[2] M.Hadac, S.Herr and H.Koch, *Well-posedness and scattering for the KP-II equation in a critical space,* Ann. Inst. H. Poincare Anal. Non lineaie. **26**(2009), no.3, 917–941.