Well-posedness for a system of quadratic derivative nonlinear Schrödinger equations at the scaling critical regularity

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We consider the Cauchy problem of the system of Schrödinger equations:

\[
\begin{aligned}
(i\partial_t + \alpha \Delta) u &= -(\nabla \cdot w)v, \quad t > 0, \quad x \in \mathbb{R}^d, \\
(i\partial_t + \beta \Delta) v &= -(\nabla \cdot w)u, \quad t > 0, \quad x \in \mathbb{R}^d, \\
(i\partial_t + \gamma \Delta) w &= \nabla (u \cdot \tau), \quad t > 0, \quad x \in \mathbb{R}^d \\
(u, v, w)|_{t=0} &= (u_0, v_0, w_0) \in H^s(\mathbb{R}^d) \times H^s(\mathbb{R}^d) \times H^s(\mathbb{R}^d)
\end{aligned}
\]

Where \(\alpha, \beta, \gamma \neq 0\) and unknown functions \(u, v, w\) are \(\mathbb{C}^d\) value. System (SS) is introduced by M.Colin and T.Colin([1]) as a model of laser-plasma interaction. M.Colin and T.Colin([1]) also proved the local existence of (SS) for \(s > d/2 + 3\). Our purpose is to improve their result and to prove the well-posedness of (SS) in the scaling critical Sobolev space.

System (SS) is invariant under the following scaling transformation:

\[ A_\lambda(t, x) = \lambda^{-1} A(\lambda^{-2}t, \lambda^{-1}x) \quad (A = (u, v, w)) \]

We note that

\[ \|A_\lambda(0, \cdot)\|_{H^s} = \lambda^{d/2-1-s}\|A(0, \cdot)\|_{H^s} \]

and the scaling critical regularity of (SS) is \(s = s_c := d/2 - 1\).

We put \(\phi := (\alpha - \gamma)(\beta + \gamma)\) and \(\theta := \alpha\beta\gamma(1/\alpha - 1/\beta - 1/\gamma)\). Main results in this talk are the following.

**Theorem 1** (For the critical result). We assume \(\phi \neq 0\).

1. If \(d \geq 4\), then (SS) is globally well-posed for small data in \(H^{s_c}\).
2. If \(d = 2, 3\) and \(\theta > 0\), then (SS) is globally well-posed for small data in \(H^{s_c}\).

**Theorem 2** (For the subcritical result). We assume \(\phi \neq 0\).

1. If \(d = 1\) and \(\theta > 0\), then (SS) is locally well-posed in \(L^2\).
2. If \(d = 1, 2, 3\) and \(\theta = 0\), then (SS) is locally well-posed in \(H^1\).
3. If \(d = 2, 3\) and \(\theta < 0\), then (SS) is locally well-posed in \(H^{1/2 + \epsilon}\) for any \(\epsilon > 0\).
4. If \(d = 1\) and \(\theta < 0\), then (SS) is locally well-posed in \(H^{1/2}\).

**Theorem 3** (Negative result). Let \(d \geq 1\).

1. If \(\phi = 0\), then flow map of (SS) is not \(C^2\) in \(H^s\) for any \(s \in \mathbb{R}\).
2. If \(\theta = 0\), then flow map of (SS) is not \(C^2\) in \(H^s\) for any \(s < 1\).
3. If \(\theta < 0\), then flow map of (SS) is not \(C^2\) in \(H^s\) for any \(s < 1/2\).

**Remark 4.** System (SS) has the following conservation quantities:

\[
M(u, v, w) := 2\|u\|_{L^2}^2 + \|v\|^2_{L^2} + \|w\|^2_{L^2},
\]

\[
H(u, v, w) := \alpha\|\nabla u\|^2_{L^2} + \beta\|\nabla v\|^2_{L^2} + \gamma\|\nabla w\|^2_{L^2} + 2\text{Re}(w, \nabla (u \cdot \tau))_{L^2}.
\]

By using the conservation law for \(M\), we can extend the local \(L^2\) solution of Theorem 2 globally in time. Furthermore, if \(\alpha, \beta\) and \(\gamma\) are same sign, then we can extend the local \(H^1\) solution of Theorem 2 globally in time by using the conservation law for \(H\).
\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|}
\hline
$\phi \neq 0$ & $d = 1$ & $d = 2,3$ & $d \geq 4$ \\
\hline
$\theta > 0$ & WP for $s \geq 0$ & WP for $s \geq s_c$ & WP for $s \geq s_c$ \\
\hline
$\theta = 0$ & WP for $s \geq 1$ & not $C^2$ for $s < 1$ & WP for $s \geq 1/2$ \\
\hline
$\theta < 0$ & WP for $s \geq 1/2$ & WP for $s > 1/2$ & not $C^2$ for $s < 1/2$ \\
\hline
$\phi = 0$ & & not $C^2$ for any $s \in \mathbb{R}$ & \\
\hline
\end{tabular}
\caption{Well-posedness (WP for short) for above the scaling critical regularity}
\end{table}

The difficulty is that there is a derivative loss arising from the nonlinear terms. To recover the derivative loss completely, we use the $U^2$, $V^2$ type Bourgain spaces which are applied to prove the well-posedness of KP-II equation in the scaling critical Sobolev space by M. Hadac, S. Herr and H. Koch (\cite{2}).

To introduce the $U^p$ and $V^p$ spaces, we define the set of finite partitions $Z$ as

$$Z := \{ \{ t_k \}_{k=0}^{K} | K \in \mathbb{N}, -\infty = t_0 < t_1 < \cdots < t_K = \infty \}.$$

**Definition 5** ($U^p$ space). Let $1 \leq p < \infty$. For $\{ t_k \}_{k=0}^{K} \in Z$ and $\{ \phi_k \}_{k=0}^{K-1} \subset L^2$ with $\sum_{k=0}^{K-1} ||\phi_k||_{L^2}^p = 1$ and $\phi_0 = 0$ we call the function $a: \mathbb{R} \rightarrow L^2$ given by

$$a(t) = \sum_{k=1}^{K} \mathbb{1}_{[t_{k-1},t_k)}(t)\phi_{k-1}$$

a “$U^p$-atom”. Furthermore, we define the atomic space

$$U^p := \left\{ u = \sum_{j=1}^{\infty} \lambda_j a_j \in L^\infty(\mathbb{R}; L^2) \mid a_j : U^p\text{-atom, } \lambda_j \in \mathbb{C} \text{ such that } \sum_{j=1}^{\infty} |\lambda_j| < \infty \right\}$$

with norm

$$||u||_{U^p} := \inf \left\{ \sum_{j=1}^{\infty} |\lambda_j| \mid u = \sum_{j=1}^{\infty} \lambda_j a_j, \ a_j : U^p\text{-atom, } \lambda_j \in \mathbb{C} \right\}.$$  

**Definition 6** ($V^p$ space). Let $1 \leq p < \infty$. We define the space of bounded $p$-variation

$$V^p := \{ v: \mathbb{R} \rightarrow L^2 \mid \lim_{t \rightarrow -\infty} v(t) \text{ and } \lim_{t \rightarrow \infty} v(t) \text{ exist, } ||v||_{V^p} < \infty \}$$

with norm

$$||v||_{V^p} := \sup_{\{ t_k \}_{k=0}^{K} \in Z} \left( \sum_{k=1}^{K} ||v(t_k) - v(t_{k-1})||_{L^2}^p \right)^{1/p},$$

where $v(-\infty) := \lim_{t \rightarrow -\infty} v(t)$ and $v(\infty) := 0$. Likewise, let $V^p_{-rc}$ denote the closed subspace of all right-continuous functions $v \in V^p$ with $\lim_{t \rightarrow -\infty} v(t) = 0$.

**Definition 7** ($U^2$, $V^2$ type Bourgain spaces). Let $s, \sigma \in \mathbb{R}$. We define the function space $Z^s_\sigma$ as the closure of all $u \in C(\mathbb{R}; H^s(\mathbb{R}^d)) \cap U^2_\sigma$ such that

$$||u||_{Z^s_\sigma} := \left( \sum_{N \geq 1} N^{2s} ||P_N u||^2_{U^2_\sigma} \right)^{1/2} < \infty$$

with respect to the $|| \cdot ||_{Z^s_\sigma}$-norm. Where

$$U^p_\sigma := \{ u: \mathbb{R} \rightarrow L^2 \mid e^{-i\sigma \Delta} u \in U^p \}, \ ||u||_{U^p_\sigma} := ||e^{-i\sigma \Delta} u||_{U^p}$$

and $P_N$ is (inhomogeneous) Littlewood-Paley decomposition operator with $x$. We also define by $Y^s_\sigma$ corresponding space where $U^2$ is replaced by $V^2_{-rc}$.
Key estimates to prove the critical result are the following.

**Proposition 8 (Key estimates).** We define

\[
I_{T,\sigma}^{(1)}(f, g)(t) := \int_0^t 1_{[0, T)}(t') e^{i(t-t')\sigma} (\nabla \cdot f(t')) g(t') dt',
\]

\[
I_{T,\sigma}^{(2)}(f, g)(t) := \int_0^t 1_{[0, T)}(t') e^{i(t-t')\sigma} \nabla (f(t') \cdot g(t')) dt',
\]

and assume \((\alpha - \gamma)(\beta + \gamma) \neq 0\).

1. If \(d \geq 4\), then for any \(0 < T < \infty\) we have
   \[
   \|I_{T,\alpha}^{(1)}(w, v)\|_{Z^{sc}_\gamma} \lesssim \|w\|_{Y^{sc}_\gamma} \|v\|_{Y^{sc}_\gamma},
   \]
   \[
   \|I_{T,\beta}^{(1)}(w, u)\|_{Z^{sc}_\gamma} \lesssim \|w\|_{Y^{sc}_\gamma} \|u\|_{Y^{sc}_\gamma},
   \]
   \[
   \|I_{T,\gamma}^{(2)}(w, v)\|_{Z^{sc}_\gamma} \lesssim \|w\|_{Y^{sc}_\gamma} \|v\|_{Y^{sc}_\gamma}.
   \]

2. If \(d = 2, 3\) and \(\alpha \beta \gamma (1/\alpha - 1/\beta - 1/\gamma) > 0\), then for any \(0 < T \leq 1\) above estimates hold.

We will talk about the properties of \(U^p, V^p\) and the outline of the proof of Proposition 8.

**REFERENCES**
