線形弾性体方程式の解の特異性について

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Theory of partial differential equations has been thoroughly developed mainly in smooth domains. In non-smooth domains such as polyhedral or cracked domains, difficulties appear because domains have singular points. In order to analyze fracture phenomena, it is important to investigate the precise behavior of the solution of the corresponding boundary value problems near singular points. Until now, we derived some convergent series expansion of solutions of boundary value problems at a crack tip and a tip of thin rigid inclusion, explicitly, ([2], [3], [5], [6]). Further they clarified the relation between the order of singularities in expansions and boundary conditions. And their analysis have possibility of application in various fields of science and engineering such as fracture problems, inverse problems (nondestructive evaluation) and so on.

In this talk we consider a reconstruction problem of an unknown polygonal cavity in a linearized elastic body [4]. In both states of plane stress and plane strain, an extraction formula of the convex hull of the unknown cavity is established by means of the *enclosure method* introduced by Ikehata [1].

Let Ω be a bounded domain of $\mathbb{R}^2$ with Lipschitz boundary and represent a homogeneous isotropic linearized elastic plate. Let D denote cavities such that $\overline{D} \subset \Omega$ and $\Omega \setminus \overline{D}$ is connected. As a priori assumption for unknown D we require that D is *polygonal* which means $D = D_1 \cup D_2 \cup \cdots \cup D_m$, $\overline{D_j} \cap \overline{D_k} = \emptyset$ for $j \neq k$, where each D_j is a simply connected open set and polygon.

Problem . *Reconstruct unknown polygonal D from a single set of a surface force and the corresponding displacement field on the boundary of Ω.*

To state our main result for the problem, we need some definitions and notations. Let $\boldsymbol{n}$ be the unit outward normal vector to $\partial(\Omega \setminus \overline{D})$. Let the displacement vector $\boldsymbol{u} = (u_1, u_2)^{\mathrm{T}} \in \{H^1(\Omega \setminus \overline{D})\}^2$ satisfy the Navier equation in the absence of any body forces

$$A\boldsymbol{u} := \frac{\tilde{E}}{2(1+\tilde{\nu})}\triangle \boldsymbol{u} + \frac{\tilde{E}}{2(1-\tilde{\nu})}\nabla(\nabla \cdot \boldsymbol{u}) = \boldsymbol{0} \quad \text{in} \quad \Omega \setminus \overline{D}$$

and the free traction condition $T\boldsymbol{u} = \boldsymbol{0}$ on ∂D where $T\boldsymbol{u}$ is the stress vector expressed by

$$T\boldsymbol{u} = \frac{\tilde{\nu}\tilde{E}}{1-\tilde{\nu}^2}(\nabla \cdot \boldsymbol{u})\boldsymbol{n} + \frac{\tilde{E}}{2(1+\tilde{\nu})}\{\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^{\mathrm{T}}\}\boldsymbol{n}$$

with

$$\tilde{E} = \begin{cases} E & \text{(plane stress)}, \\ \dfrac{E}{1-\nu^2} & \text{(plane strain)}, \end{cases} \qquad \tilde{\nu} = \begin{cases} \nu & \text{(plane stress)}, \\ \dfrac{\nu}{1-\nu} & \text{(plain strain)}. \end{cases}$$

Here E and ν are Young's modulus and Poisson's ratio of the elastic medium, respectively. Since both the shear modulus and the bulk modulus are required to be positive, we suppose $E > 0$ and $-1 < \nu < \frac{1}{2}$.

We denote by h_D the support function of D:

$$h_D(\boldsymbol{\omega}) = \sup_{\boldsymbol{x} \in D} \boldsymbol{x} \cdot \boldsymbol{\omega} \quad \text{for} \quad \boldsymbol{\omega} \in S^1.$$

Next, we introduce the following assumptions for $\boldsymbol{\omega}$.

(A) $\boldsymbol{\omega}$ satisfies that the intersection of the line $\boldsymbol{x} \cdot \boldsymbol{\omega} = h_D(\boldsymbol{\omega})$ with ∂D consists of only one point,

(A') $\boldsymbol{\omega}$ satisfies (A) and that the interior angle bisector of D at the point $\{\boldsymbol{x} \in \mathbb{R}^2 \mid \boldsymbol{x} \cdot \boldsymbol{\omega} = h_D(\boldsymbol{\omega})\} \cap \partial D$ is not perpendicular to the line $\boldsymbol{x} \cdot \boldsymbol{\omega} = h_D(\boldsymbol{\omega})$.

Then our main result is the following formula:

THEOREM ([4]). *Let* $\boldsymbol{u}$ *be not a rigid displacement and* D *be polygonal. Under the assumption of* (A') *the formula*

$$h_D(\boldsymbol{\omega}) = \lim_{\tau \longrightarrow \infty} \frac{1}{\tau} \log \left| \int_{\partial\Omega} (T\boldsymbol{u} \cdot \boldsymbol{v} - T\boldsymbol{v} \cdot \boldsymbol{u}) \, \mathrm{d}S \right|,$$

is valid, where $\boldsymbol{v}(\boldsymbol{x}) = (\boldsymbol{\omega} + i\boldsymbol{\omega}^{\perp})e^{\tau \boldsymbol{x}\cdot(\boldsymbol{\omega}+i\boldsymbol{\omega}^{\perp})}$, $\tau > 0$; $\boldsymbol{\omega}^{\perp} \in S^1$ *is perpendicular to* $\boldsymbol{\omega}$ *and satisfies* $\det(\boldsymbol{\omega}^{\perp} \ \boldsymbol{\omega}) > 0$.

Some remarks on Theorem are in order.

- For given polygonal D the set of all $\boldsymbol{\omega}$ that does not satisfy (A') (or (A)) is a finite set. Since the support function is continuous, it follows from Theorem that a single set of the data $\boldsymbol{u}$, $T\boldsymbol{u}$ on $\partial\Omega$ uniquely determine $h_D(\boldsymbol{\omega})$ for all $\boldsymbol{\omega} \in S^1$ and thus the convex hull of D via the formula

$$\bigcap_{\boldsymbol{\omega}\in S^1} \{\boldsymbol{x} \in \mathbb{R}^2 \mid \boldsymbol{x}\cdot\boldsymbol{\omega} < h_D(\boldsymbol{\omega})\}.$$

  This is the origin of the name "the enclosure method" and the solution of our problem.

- In contrast to the related results (e.g. [1]), we do not require any other a priori assumptions for D excepting "polygonal" and any constraints on boundary data. This reflects the behavior of solutions of the boundary value problem near a vertex.

- If we assume (A) instead of (A'), then the formula is valid by replacing $\lim_{\tau\longrightarrow\infty}$ with $\limsup_{\tau\longrightarrow\infty}$.

- In two-dimensional case rigid displacements can be described in the form $F(\boldsymbol{x})\boldsymbol{k} = (k_0 + k_2x_2, k_1 - k_2x_1)^{\mathrm{T}}$ with an arbitrary constant vector $\boldsymbol{k} = (k_0, k_1, k_2)^{\mathrm{T}}$.

This result is based on a joint work with Masaru Ikehata(Hiroshima University) [4].

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