Leray's problem on D-solutions to the stationary Navier-Stokes equations past an obstacle

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We consider the stationary problem of the Navier-Stokes equations in the 3-dimensional exterior domain Ω. For every $u_\infty \neq 0$ and every $f \in \dot{H}_2^{-1}(\Omega)$, Leray constructed a weak solution u with $\nabla u \in L_2(\Omega)$ and $u - u_\infty \in L_6(\Omega)$. Here $\dot{H}_2^{-1}(\Omega)$ denotes the dual space of the homogeneous Sobolev space $\dot{H}_2^1(\Omega)$. Such a weak solution u is called a $\mathcal{D}$-solution. We prove that every $\mathcal{D}$-solution u fulfills the additional regularity property $u - u_\infty \in L_4(\Omega)$ and $u_\infty \cdot \nabla u \in \dot{H}_2^{-1}(\Omega)$ without any restriction on f except for $f \in \dot{H}_2^{-1}(\Omega)$. As a consequence, it turns out that every $\mathcal{D}$-solution necessarily satisfies the generalized energy equality. Moreover, we obtain a sharp a priori estimate and uniqueness result for $\mathcal{D}$-solutions assuming only that $\|f\|_{\dot{H}_2^{-1}(\Omega)}$ and $|u_\infty|$ are suitably small. Our results give final affirmative answers to open questions proposed by Leray on the energy equality and uniqueness of $\mathcal{D}$ solutions. Finally, we investigate the convergence of weak solutions as $u_\infty \to 0$ in the strong norm topology, while the limiting weak solution exhibits a completely different behavior from that in the case $u_\infty \neq 0$.