Time-global solutions of the degenerate Keller-Segel system in higher dimensions

Yoshifumi MIMURA
Tokyo University of Sciences

In this talk I will discuss the time-global existence of the solution of the following degenerate Keller-Segel system in higher dimensions $d > 2$.

$$\begin{cases} \partial_t u = \nabla \cdot (\nabla u^m - \chi u \nabla v) & \text{in } \Omega,\ t > 0, \\ \varepsilon \partial_t v = \Delta v - \gamma v + \alpha u & \text{in } \Omega,\ t > 0, \\ u(x,0) = u_0(x) \geq 0, & x \in \Omega, \\ \varepsilon v(x,0) = \varepsilon v_0(x) \geq 0, & x \in \Omega, \end{cases} \tag{KS}$$

where Ω is a bounded domain of $\mathbb{R}^d$ with smooth boundary, $\alpha, \chi, \varepsilon$ are positive constants and γ is non-negative constant. Besides, we focus on the case $m = 2 - 2/d$ and impose the following boundary conditions:

$$\frac{\partial u^m}{\partial \boldsymbol{\nu}} - \chi u \frac{\partial v}{\partial \boldsymbol{\nu}} = v = 0 \quad \text{on } \partial\Omega,\ t > 0. \tag{BC}$$

The above system is an extension of the model that was originally proposed by Keller and Segel in 1970 to describe the aggregation of slime molds. In the original Keller-Segel system, in which $m = 1$ and $d = 2$, it is expected that there exists the following threshold mass M_c:

> If $0 < \|u_0\|_{L^1} < M_c$, then the solutions of (KS) exist globally in time, while for any $M > M_c$ there exists a solution with $\|u_0\|_{L^1} = M$ that blow up in finite time.

This phenomenon is known to be true if $\varepsilon = 0$. However, for $\varepsilon > 0$, only partial results are known. In particular, the existence of blow-up for any $M > M_c$ is not known yet for $\varepsilon > 0$ (at least no proof has been published).

One of the fundamental questions concerning the system (KS) is whether or not a threshold mass M_c such as above exists in the higher dimensional case $d > 2$. Note that, because of a certain scaling property of (KS), it is not difficult to see that a threshold mass can possibly exist only if $m = 2 - 2/d$, and this is why we are focusing on this case.

In the parabolic-elliptic system (where $\varepsilon = 0$) with $m = 2 - 2/d, d > 2$, Blanchet *et al.*(2009) have proved the existence of the threshold mass M_c. In this talk, I will show that the solutions of the parabolic-parabolic system (KS) (where $\varepsilon > 0$) exist globally in time if $\|u_0\|_{L^1} < M_*$, where M_* is a certain critical value associated with the Lyapunov functional for (KS). It turns out that this constant M_* coincides with the threshold mass M_c of Blanchet *et al.*

The proof relies on the interpretation of (KS) as a gradient flow in $\mathscr{P}_2(\Omega) \times L^2(\Omega)$, where $\mathscr{P}_2(\Omega)$ denotes the set of probability measures with finite quadratic moment endowed with the Wasserstein distance. More precisely, we first construct time-discretized solutions that are given by solving a certain minimizing problem at each time step, and show that under the condition $\|u_0\| < M_*$, the time-discretized solutions converge to a weak solution of (KS) as the time step size tends to zero.