# Stability of line solitons for KP-II

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The KP-II equation

$$\partial_x(\partial_t u + \partial_x^3 u + 3\partial_x(u^2)) + 3\sigma\partial_y^2 u = 0 \quad \text{for } t > 0 \text{ and } (x, y) \in \mathbb{R}^2 \tag{KP}$$

is a 2-dimensional generalization of the KdV equation

$$\partial_t u + \partial_x^3 u + 3\partial_x(u^2) = 0 \quad \text{for } t > 0 \text{ and } x \in \mathbb{R}. \tag{1}$$

The KP equation takes slow variations in the transversal direction into account. If $\sigma = 1$, then (KP) is called KP-II and it describes the motion of shallow water waves with weak surface tension. Obviously, any solutions of (1) satisfy (KP).

The KdV equation has a two parameter family of solitary wave solutions

$$\{\varphi_c(x - 2ct - \gamma) \mid c > 0\,, \gamma \in \mathbb{R}\}\,, \quad \varphi_c(x) = c\,\mathrm{sech}^2\left(\sqrt{\frac{c}{2}}\,x\right).$$

The KP-II equation is supposed to explain stability of these solitary wave solutions to the transversal perturbations (see [7]). When $\varphi_c(x - 2ct)$ is considered as a solution of the KP equations, we call it a *line soliton*. In this talk, I will discuss stability of line soliton solutions for the KP-II equation. The KP-II equation is known to be well-posed in $L^2(\mathbb{R}^2)$ on the background of line solitons ([11]).

The KP equation (KP) is an integrable system as well as the KdV equation and have conserved quantities such as

$$\int_{\mathbb{R}^2} u(t, x, y)^2\, dxdy \quad \text{(momentum)},$$

$$\frac{1}{2}\int \left(u_x^2(t, x, y) - 3\sigma(\partial_x^{-1}\partial_y u(t, x, y))^2 - 2u^3(t, x, y)\right) dxdy \quad \text{(Hamiltonian)}.$$

For the KP-I equation, that is (KP) with $\sigma = -1$, the first two terms of the Hamiltonian have the same sign and it was shown by [6, 14] that there exists a stable ground state for the KP-I equation, whereas the KP-II equation has no traveling wave solutions that belong to $L^2(\mathbb{R}^2)$ ([5]).

On the other hand, it is known that the line solitons of the KP-I equation are unstable ([16]) and line solitons of the KP-II equation are linearly stable. Because the dominant quadratic part of the Hamiltonian of the KP-II equation is indefinite, it is more natural to explain stability of line solitons by using propagation estimates such as [13] rather than by using variational arguments such as [3].

The main difference between stability analysis of the KdV 1-soliton or the stability of line soliton with the $y$-periodic boundary condition is that $\varphi_c(x - 2ct)$ does not have the finite $L^2$-mass because it is not localized in the $y$-direction. As a consequence, the linearized operator of the KP-II equation around the line soliton has a family of continuous eigenvalues converging to 0 (see e.g. [1, 2]) in exponentially weighted space, whereas 0 is an isolated eigenvalue of the linearized KdV operator around a solitary wave in exponentially weighted space.

Since we have continuous spectrum converging to 0, the modulations of the speed and the phase shift of line solitons for the KP-II equation cannot be described by ODEs as KdV ([13]) or the KP-II equation posed on $\mathbb{R}_x \times \mathbb{R}_y/(2\pi\mathbb{Z})$ ([10]).

I find that for the KP-II equation posed on $\mathbb{R}^2$, modulation of the speed parameter $c(t,y)$ and the phase shift $x(t,y)$ cannot be uniform in $y$ and their long time behavior is described by the Burgers equation. Note that similar modulation equations have formally derived for a two spatial dimensional Boussinesq model ([12]).

My results are the following ([9]).

**Theorem 1** *Let* $c_0 > 0$ *and* $a \in (0, \sqrt{c_0/2})$. *Then there exist positive constants* $\varepsilon_0$ *and* $C$ *satisfying the following: if* $u(0,x,y) = \varphi_{c_0}(x) + v_0(x,y)$, $v_0 \in H^1(\mathbb{R}^2)$ *and* $\varepsilon := \|e^{ax}v_0\|_{L^2(\mathbb{R}^2)} + \|e^{ax}v_0\|_{L^1_yL^2_x} + \|v_0\|_{L^2(\mathbb{R}^2)} < \varepsilon_0$, *then there exist* $C^1$*-functions* $c(t,y)$ *and* $x(t,y)$ *such that for* $t \geq 0$,

$$\|u(t,x,y) - \varphi_{c(t,y)}(x - x(t,y))\|_{L^2(\mathbb{R}^2)} \leq C\varepsilon\,, \tag{2}$$

$$\sup_{y\in\mathbb{R}}(|c(t,y) - c_0| + |x_y(t,y)|) \leq C\varepsilon(1+t)^{-1/2}\,, \tag{3}$$

$$\|x_t(t,\cdot) - 2c(t,\cdot)\|_{L^2} \leq C\varepsilon(1+t)^{-3/4}\,, \tag{4}$$

$$\left\|e^{ax}(u(t,x+x(t,y),y) - \varphi_{c(t,y)}(x))\right\|_{L^2} \leq C\varepsilon(1+t)^{-3/4}\,. \tag{5}$$

We find that $c(t,y)$ and $\partial_y x(t,y)$ behave like a self-similar solution of the Burgers equation around $y = \pm\sqrt{8c_0}t$.

**Theorem 2** *Let* $c_0 = 2$ *and let* $v_0$ *and* $\varepsilon$ *be the same as in Theorem 1. Then for any* $R > 0$,

$$\left\|\begin{pmatrix} c(t,\cdot) \\ x_y(t,\cdot)\end{pmatrix} - \begin{pmatrix} 2 & 2 \\ 1 & -1\end{pmatrix}\begin{pmatrix} u_B^+(t,y+4t) \\ u_B^-(t,y-4t)\end{pmatrix}\right\|_{L^2(|y\pm4t|\leq R\sqrt{t})} = o(t^{-1/4})$$

*as* $t \to \infty$, *where* $u_B^\pm$ *are self similar solutions of the Burgers equation*

$$\partial_t u = 2\partial_y^2 u \pm 4\partial_y(u^2)$$

*such that*

$$u_B^\pm(t,y) = \frac{\pm m_\pm H_{2t}(y)}{2\left(1 + m_\pm\int_0^y H_{2t}(y_1)\,dy_1\right)}\,, \quad H_t(y) = (4\pi t)^{-1/2}e^{-y^2/4t}\,,$$

*and that* $m_\pm$ *are constants satisfying*

$$\int_{\mathbb{R}} u_B^\pm(t,y)\,dy = \frac{1}{4}\int_{\mathbb{R}} c(0,y)\,dy + O(\varepsilon^2)\,.$$

The KP-II equation (KP) is invariant under a change of variables

$$x \mapsto x + ky - 3k^2t + \gamma \quad \text{and} \quad y \mapsto y - 6kt \quad \text{for any } k,\, \gamma \in \mathbb{R}, \tag{6}$$

and has a 3-parameter family of line soliton solutions

$$\mathcal{A} = \{\varphi_c(x + ky - (2c + 3k^2)t + \gamma) \mid c > 0\,,\ k\,, \gamma \in \mathbb{R}\}\,.$$

Thanks to propagations of the local phase shifts along the crest of line solitons, the set $\mathcal{A}$ is not stable in $L^2(\mathbb{R}^2)$.

**Theorem 3** *Let $c_0 > 0$. There exists a positive constant $C$ such that for any $\varepsilon > 0$, there exists a solution of* (KP) *such that $\|u(0,x,y) - \varphi_{c_0}(x)\|_{L^2} < \varepsilon$ and*

$$\liminf_{t\to\infty} t^{-1/4} \inf_{v\in\mathcal{A}} \|u(t,x,y) - v\|_{L^2(\mathbb{R}^2)} \geq C\varepsilon\,.$$

Similar problems have been studied for nonlinear heat equations ([15, 8]) and also for kink solutions to a wave equation ([4]). However, we need some correction of the phase shift for the KP-II equation, which does not appear in these former results because the transversal dimension and the power of nonlinear terms are higher than our problem.

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