On the number of peaks of the eigenfunctions of the linearized Gel'fand problem [1]

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This is a joint work with Francesca Gladiali (Università degli Studi di Sassari) and Massimo Grossi (Università di Roma "La Sapienza") [GGOS13].

In this talk we consider the Gel'fand problem,

$$-\Delta u = \lambda e^u \ \text{ in } \Omega, \quad u = 0 \ \text{ on } \partial\Omega, \tag{1}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary $\partial\Omega$ and $\lambda > 0$ is a real parameter. Let $\{\lambda_n\}_{n\in\mathbb{N}}$ be a sequence of positive values such that $\lambda_n \to 0$ as $n \to \infty$ and let $u_n = u_n(x)$ be a sequence of solutions of (1) for $\lambda = \lambda_n$. In [NS90], the authors studied solutions $\{u_n\}$ which blow-up at m-points, that is, there is a set $\mathcal{S} = \{\kappa_1, \cdots, \kappa_m\} \subset \Omega$ of m distinct points such that $\|u_n\|_{L^\infty(\omega)} = O(1)$ for any $\omega \Subset \overline{\Omega} \setminus \mathcal{S}$ and $u_n|_{\mathcal{S}} \to +\infty$ as $n \to \infty$, see [EGP05] or [DKM05] for some sufficient conditions which ensure the existence of this type of solutions. In this talk we consider solutions u_n to (1) with m blow-up points and investigate the eigenvalue problem

$$-\Delta v_n^k = \mu_n^k \lambda_n e^{u_n} v_n^k \ \text{ in } \Omega, \quad v_n^k = 0 \ \text{ on } \partial\Omega, \quad \|v_n^k\|_\infty = \max_{\overline{\Omega}} v_n^k = 1, \tag{2}$$

which admits a sequence of eigenvalues $\mu_n^1 < \mu_n^2 \le \mu_n^3 \le \dots$, where v_n^k is the k-th eigenfunction of (2) corresponding to the eigenvalue μ_n^k.

One of the main results concerns pointwise estimates of the eigenfunction. In particular, we are interested in *the number of peaks* of v_n^k for $k = 1, .., m$. In our previous work [GGOS12], we have that

$$v_n^k \to 0 \quad \text{in } C^1\left(\overline{\Omega} \setminus \cup_{j=1}^m B_R(\kappa_j)\right)$$

for some fixed $0 < R << 1$. This means that v_n^k can concentrate only at κ_j, $j = 1, .., m$. This leads to the following definition,

Definition 1. *We say that an eigenfunction v_n^k concentrates at $\kappa_j \in \Omega$ if there exists $\kappa_{j,n} \to \kappa_j$ such that*

$$\left|v_n^k(\kappa_{j,n})\right| \ge C > 0 \quad \text{for } n \text{ large.} \tag{3}$$

A problem that arises naturally is the following:

Question 1. *Let us suppose that u_n blows-up at the points $\{k_1, .., k_m\}$. Is the same true for the eigenfunction v_n^k, $k =, 1.., m$ associated to a simple eigenvalue μ_n^k of (2)?*

Obviously if the eigenvalue μ_n^k is multiple, in general it makes no sense to speak about the number of point of concentration, since this depends on the linear combination of the eigenfunctions.

A first partial answer related to this question was given in [GGOS12], where the following result was proved.

Theorem 2. *For each $k \in \{1, \dots, m\}$ there exists a vector*

$$\mathfrak{c}^k = (c_1^k, \dots, c_m^k) \in [-1, 1]^m \subset \mathbb{R}^m, \quad \mathfrak{c}^k \neq \mathbf{0} \tag{4}$$

such that for each $j \in \{1, \dots, m\}$, there exists a subsequence satisfying

$$\frac{v_n^k}{\mu_n^k} \to 8\pi \sum_{j=1}^m c_j^k \, G(\cdot, \kappa_j) \quad \text{in } C^{2,\alpha}_{loc}\left(\overline{\Omega} \setminus \{\kappa_1, \dots, \kappa_m\}\right). \tag{5}$$

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Here $G(x,y)$ denotes the Green function of $-\Delta$ in Ω with Dirichlet boundary condition, $K(x,y) = G(x,y) - \frac{1}{2\pi}\log|x-y|^{-1}$ the regular part of $G(x,y)$, and $R(x) = K(x,x)$ the Robin function. A consequence of Theorem 2 is that

$$v_n^k \text{ concentrates at } \kappa_j \text{ if and only if } c_j^k \neq 0. \tag{6}$$

We characterize the values c_j^k in term of the Green function and this will allow us to determine whether c_j^k is equal to 0 or not.

Theorem 3. *For each $k \in \{1,\dots,m\}$, we have that $\mathfrak{c}^k = (c_1^k,\dots,c_m^k) \in [-1,1]^m \subset \mathbb{R}^m \setminus \{\mathbf{0}\}$ is the k-th eigenvector of the matrix (h_{ij}) given by*

$$h_{ij} = \begin{cases} R(\kappa_i) + 2\sum_{\substack{1\le h\le m\\ h\neq i}} G(\kappa_h,\kappa_i) & \text{if } i=j,\\ -G(\kappa_i,\kappa_j) & \text{if } i\neq j. \end{cases} \tag{7}$$

From Theorem 3 we can deduce the answer to the Question 1,

Corollary 4. *Let $\mathfrak{c}^k = (c_1^k,..,c_m^k)$ be the k-th eigenvector of the matrix (h_{ij}). Then if μ_n^k is simple and if $c_j^k \neq 0$ we have that v_n^k concentrates at k_j.*

Our next aim is to understand better when $c_j^k \neq 0$. The following proposition gives some information in this direction.

Theorem 5. *Let $k \in \{1,\dots,m\}$, μ_n^k a simple eigenvalue and v_n^k the corresponding eigenfunction. Then we have that,*
i) any v_n^1 concentrates at m points $\kappa_1,..,\ \kappa_m$,
ii) any v_n^k concentrates at least at two points κ_i, κ_j with $i,j \in \{1,..,m\}$.

The previous results are a consequence of the following theorem, which is a refinement up the second order of some estimates proved of [GGOS12].

Theorem 6. *For each $k \in \{1,\dots,m\}$, it holds that*

$$\mu_n^k = -\frac{1}{2}\frac{1}{\log\lambda_n} + \left(2\pi\Lambda^k - \frac{3\log 2 - 1}{2}\right)\frac{1}{(\log\lambda_n)^2} + o\left(\frac{1}{(\log\lambda_n)^2}\right) \tag{8}$$

as $n \to +\infty$, where Λ^k is the k-th eigenvalue of the $m \times m$ matrix (h_{ij}) defined in (7) assuming $\Lambda^1 \le \dots \le \Lambda^m$.

So the effect of the domain Ω on the eigenvalues μ_n^k appears in the second order term of the expansion of μ_n^k.

References

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