Solvability of the initial value problem to a model system for water waves

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The water wave problem is mathematically formulated as a free boundary problem for an irrotational flow of an inviscid and incompressible fluid under the gravitational field. The basic equations for water waves are complicated due to the nonlinearity of the equations together with the presence of an unknown free surface. Therefore, until now many approximate equations have been proposed and analyzed to understand natural phenomena for water waves. Famous examples of such approximate equations are the shallow water equations, the Green–Naghdi equations, Boussinesq type equations, the Korteweg–de Vries equation, the Kadomtsev–Petviashvili equation, the Benjamin–Bona–Mahony equation, the Camassa–Holm equation, the Benjamin–Ono equations, and so on. All of them are derived from the water wave problem under the shallowness assumption of the water waves, which means that the mean depth of the water is sufficiently small compared to the typical wavelength of the water surface.

On the other hand, it is well-known that the water wave problem has a variational structure. In fact, J. C. Luke (1967) gave a Lagrangian in terms of the velocity potential and the surface variation, and showed that the corresponding Euler–Lagrange equations are the basic equations for water waves. M. Isobe (1994) and T. Kakinuma (2000) derived model equations for water waves without any shallowness assumption. The model equations are the Euler–Lagrange equations to an approximated Lagrangian, which is obtained by approximating the velocity potential in Luke's Lagrangian. In this talk, we consider the initial value problem to one of the model equations

$$\begin{cases} \eta_t + \nabla\cdot\Big(H\nabla\phi^0 + \dfrac{1}{3}H^3\nabla\phi^1 - H^2\phi^1\nabla b\Big) = 0, \\ H^2\eta_t + \nabla\cdot\Big(\dfrac{1}{3}H^3\nabla\phi^0 + \dfrac{1}{5}H^5\nabla\phi^1 - \dfrac{1}{2}H^4\phi^1\nabla b\Big) \\ \quad +H^2\nabla b\cdot\nabla\phi^0 + \dfrac{1}{2}H^4\nabla b\cdot\nabla\phi^1 - \dfrac{4}{3}H^3(1+|\nabla b|^2)\phi^1 = 0, \\ \phi^0_t + H^2\phi^1_t + g\eta + \dfrac{1}{2}|\nabla\phi^0|^2 + \dfrac{1}{2}H^4|\nabla\phi^1|^2 \\ \quad +H^2\nabla\phi^0\cdot\nabla\phi^1 - 2H\phi^1\nabla b\cdot\nabla\phi^0 - 2H^3\phi^1\nabla b\cdot\nabla\phi^1 + 2H^2(1+|\nabla b|^2)(\phi^1)^2 = 0 \end{cases}$$

under the initial conditions

$$(\eta, \phi^0, \phi^1) = (\eta_0, \phi^0_0, \phi^1_0) \qquad \text{at} \quad t=0,$$

where $\eta = \eta(x,t)$ is the surface elevation, $b = b(x)$ represents the bottom topography, $\phi^0 = \phi^0(x,t)$ and $\phi^1 = \phi^1(x,t)$ are related to the velocity potential $\Phi = \Phi(x,z,t)$ of the water by an approximate formula $\Phi(x,z,t) = \phi^0(x,t) + (z-b(x))^2\phi^1(x,t)$ and $H = H(x,t)$ is the depth of water and is given by $H(x,t) = h + \eta(x,t) - b(x)$. Here, g is the gravitational constant and h is the mean depth of the water. t is the time, $x = (x_1, x_2, \ldots, x_n) \in \mathbf{R}^n$ is the horizontal spatial coordinates, and z is the vertical spatial coordinate. The model are nonlinear dispersive equations and the hypersurface $t = 0$ is characteristic for the model equations. Therefore, the initial data have to be restricted in an infinite dimensional manifold in order to the existence of the solution, and we show that the manifold is invariant under the time evolution.

This talk is based on the joint research with my former student Yuuta Murakami.