Decomposition of the Möbius energy: Geometric meaning and analytic utility

Aya Ishizeki (Saitama University)
Nagasawa Takeyuki (Saitama University)

1 Introduction

Let $\boldsymbol{f} : \mathbb{R}/\mathcal{L}\mathbb{Z} \ni s \mapsto \boldsymbol{f}(s) \in \mathbb{R}^n$ be a closed curve in $\mathbb{R}^n$ with total length $\mathcal{L}$. s is an arc-length parameter. We denote the distance between $\boldsymbol{f}(s_1)$ and $\boldsymbol{f}(s_2)$ along the closed curve by $\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))$.

The Möbius energy $\mathcal{M}$ is defined by

$$(1) \qquad \mathcal{M}(\boldsymbol{f}) = \text{p.v.} \iint \left(\frac{1}{\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^2} - \frac{1}{\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))^2} \right) ds_1 ds_2,$$

where p.v. $\iint = \lim_{\varepsilon \to +0} \iint_{|s_1 - s_2| > \varepsilon}$ is Cauchy's principal value.

Each term of the integrand in (1) is not integrable on $(\mathbb{R}/\mathcal{L}\mathbb{Z})^2$. The subtraction of two terms gains the integrability, and therefore it is not easy to find the proper domain of $\mathcal{M}$. In this talk, we give decomposition of $\mathcal{M}$ into three parts $\mathcal{M}_1$, $\mathcal{M}_2$ and the absolute constant 4. The first part $\mathcal{M}_1$ is a positive definite functional which implies the proper domain of $\mathcal{M}$. The second one $\mathcal{M}_2$ has the determinant structure which characterizes the cancellation of the integrand.

2 The decomposition of the Möbius energy

At a point where $\boldsymbol{f}$ is differentiable, we denote the unit tangent vector by $\boldsymbol{\tau}$. Similarly $\boldsymbol{\kappa}$ stands for the curvature vector at a point where $\boldsymbol{f}$ is twice differentiable.

Theorem 1 *The energy $\mathcal{M}(\boldsymbol{f})$ is decomposed as*

$$\mathcal{M}(\boldsymbol{f}) = \iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \left(\mathscr{M}_1(\boldsymbol{f}) + \mathscr{M}_2(\boldsymbol{f}) \right) ds_1 ds_2 + 4,$$

where

$$\mathscr{M}_1(\boldsymbol{f}) = \frac{\|\boldsymbol{\tau}(s_1) - \boldsymbol{\tau}(s_2)\|_{\mathbb{R}^n}^2}{2\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^2},$$

$$\mathscr{M}_2(\boldsymbol{f}) = \frac{2}{\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^4} \times \det \begin{pmatrix} \boldsymbol{\tau}(s_1)\cdot\boldsymbol{\tau}(s_2) & (\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2))\cdot\boldsymbol{\tau}(s_1) \\ (\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2))\cdot\boldsymbol{\tau}(s_2) & \|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^2 \end{pmatrix}.$$

We replace the Euclidean distance $\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}$ in the denominator of $\mathscr{M}_1(\boldsymbol{f})$ by the distance $\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))$ along the curve. Its integration is

$$\iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \frac{\|\boldsymbol{\tau}(s_1) - \boldsymbol{\tau}(s_2)\|_{\mathbb{R}^n}^2}{2\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))^2} ds_1 ds_2 = \frac{1}{2}[\boldsymbol{\tau}]_{\frac{1}{2},2}^2,$$

where $[\cdot]_{\frac{1}{2}}$ is the intrinsic semi-norm of $H^{\frac{1}{2}}(\mathbb{R}/\mathcal{L}\mathbb{Z})$. Roughly speaking, $\mathcal{M}_1(\boldsymbol{f}) < \infty$ implies $\boldsymbol{\tau} \in H^{\frac{1}{2}}(\mathbb{R}/\mathcal{L}\mathbb{Z})$, *i.e.*, $\boldsymbol{f} \in H^{\frac{3}{2}}(\mathbb{R}/\mathcal{L}\mathbb{Z})$. This is true if $\boldsymbol{f}$ is bi-Lipschitz. It is known that $\mathcal{M}(\boldsymbol{f}) < \infty$ implies the bi-Lipschitz continuity.

Theorem 2 *Supppose that $\boldsymbol{f}$ has no self-intersections.*

1. *Assume that $\boldsymbol{f}$ is bi-Lipschitz and $\boldsymbol{f} \in H^{\frac{3}{2}}(\mathbb{R}/\mathcal{L}\mathbb{Z})$. Then we have $\mathscr{M}_i(\boldsymbol{f}) \in L^1((\mathbb{R}/\mathcal{L}\mathbb{Z})^2)$. In fact there exists a constant C depending on the bi-Lipschitz constant of $\boldsymbol{f}$ such that*

$$\|\mathscr{M}_i(\boldsymbol{f})\|_{L^1} \le C\,[\boldsymbol{\tau}]_{\frac{1}{2},2}^2 .$$

2. *Suppose that $\boldsymbol{f} \in C^{1,1}(\mathbb{R}/\mathcal{L}\mathbb{Z})$. Then $\mathscr{M}_i(\boldsymbol{f}) \in L^\infty((\mathbb{R}/\mathcal{L}\mathbb{Z})^2)$ holds.*

3. *Let $\boldsymbol{f} \in C^2(\mathbb{R}/\mathcal{L}\mathbb{Z})$ and put $\mathscr{M}_i(\boldsymbol{f})(s,s) = \dfrac{(-1)^{i-1}}{2}\|\boldsymbol{\kappa}(s)\|_{\mathbb{R}^n}^2$. Then $\mathscr{M}_i(\boldsymbol{f}) \in C((\mathbb{R}/\mathcal{L}\mathbb{Z})^2)$ holds.*

Since s is the arc-length parameter, we have $\|\boldsymbol{\tau}\|_{\mathbb{R}^n} = 1$. In the proof of the first statement we use this fact, *i.e.*, $\boldsymbol{f} \in H^{1,\infty}(\mathbb{R}/\mathcal{L}\mathbb{Z})$. As we see later the first variation of $\mathcal{M}_i$ is a linear functional on $H^{1,\infty}(\mathbb{R}/\mathcal{L}\mathbb{Z}) \cap H^{\frac{3}{2}}(\mathbb{R}/\mathcal{L}\mathbb{Z})$. Consequently we conclude that the proper domain of $\mathcal{M}$ is $H^{1,\infty}(\mathbb{R}/\mathcal{L}\mathbb{Z}) \cap H^{\frac{3}{2}}(\mathbb{R}/\mathcal{L}\mathbb{Z})$. See [1].

3 The Möbius invariance

The energy $\mathcal{M}$ is called the Möbius energy because it is invariant under the Möbius transformation. Surprisingly both $\mathcal{M}_1$ and $\mathcal{M}_2$ have the Möius invariance.

Theorem 3 *We put*

$$\mathcal{M}_i(\boldsymbol{f}) = \iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \mathscr{M}_i(\boldsymbol{f}) ds_1 ds_2.$$

Both $\mathcal{M}_1$ and $\mathcal{M}_2$ are Möbius invariance in the following sense.

1. *The energy is invariant under the dilation.*
2. *Let $\boldsymbol{f} \to \boldsymbol{p}$ be the inversion with respect to a circle with the center $\boldsymbol{c}$.*

   (1) $\mathcal{M}_1(\boldsymbol{f}) + \mathcal{M}_2(\boldsymbol{f}) = \mathcal{M}_1(\boldsymbol{p}) + \mathcal{M}_2(\boldsymbol{p})$ *holds for* $\boldsymbol{f} \in W^{1,1}(\mathbb{R}/2\pi\mathbb{Z})$, *even if $\boldsymbol{c}$ is on the closed curve $\boldsymbol{f}$.*

   (2) *Assume that $\boldsymbol{c}$ is not on $\boldsymbol{f}$. If $\mathcal{M}(\boldsymbol{f}) < \infty$, then $\mathcal{M}_1(\boldsymbol{f}) = \mathcal{M}_1(\boldsymbol{p})$ and $\mathcal{M}_2(\boldsymbol{f}) = \mathcal{M}_2(\boldsymbol{p})$ hold.*

Remark 1 *$\mathcal{M}_2$ is the same up to constant multiplication as the E_{OS}-energy which is introduced by O'Hara and Solanes [5] as a Möbius invariant energy. Theorem 3 shows the Möbius invariance of the E_{OS}-energy under a less regularity condition.*

Let S^1 be a circle in $\mathbb{R}^n$. Then it is known that $\mathcal{M}(S^1) = 4$ and it is the minimum value of $\mathcal{M}$. This shows that $\mathcal{M}_1 + \mathcal{M}_2$ is a non-negative functional. In fact the energy density $\mathscr{M}_1 + \mathscr{M}_2$ is non-negative.

Theorem 4 (Non-negativity of the energy density) *The sum of the density is non-negative. In fact it holds that*

$$\begin{aligned}
&\mathscr{M}_1(\boldsymbol{f}) + \mathscr{M}_2(\boldsymbol{f}) \\
&= \frac{1}{\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^2} \\
&\quad \times \left\{ \frac{1}{2} \|\boldsymbol{\tau}(s_1) + \boldsymbol{\tau}(s_2)\|_{\mathbb{R}^n}^2 \|\boldsymbol{\nu}_*\|_{\mathbb{R}^n}^2 + (\boldsymbol{\nu}_* \cdot \boldsymbol{\tau}(s_1))^2 + (\boldsymbol{\nu}_* \cdot \boldsymbol{\tau}(s_2))^2 \right\} \geq 0,
\end{aligned}$$

where

$$\boldsymbol{\nu}_* = \boldsymbol{\tau}_* - (\boldsymbol{\tau}_* \cdot \boldsymbol{\tau}_m)\boldsymbol{\tau}_m, \quad \boldsymbol{\tau}_* = \frac{|s_1 - s_2|(\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2))}{\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}(s_1 - s_2)}$$

and the vector $\boldsymbol{\tau}_m = \boldsymbol{\tau}_m(s_1, s_2)$ *is defined as follows. If* $\boldsymbol{\tau}(s_1) + \boldsymbol{\tau}(s_2) \neq \boldsymbol{o}$, *then we define*

$$\boldsymbol{\tau}_m = \frac{\boldsymbol{\tau}(s_1) + \boldsymbol{\tau}(s_2)}{\|\boldsymbol{\tau}(s_1) + \boldsymbol{\tau}(s_2)\|_{\mathbb{R}^n}}.$$

Otherwise $\boldsymbol{\tau}_m$ *is defined by any unit vector satisfying* $\boldsymbol{\tau}_m \cdot \boldsymbol{\tau}(s_1) = 0$.

Corollary 1 *Let* $\boldsymbol{f}$ *be a closed curve.* $\mathcal{M}(\boldsymbol{f}) = 4$ *holds if and only if* $\boldsymbol{f}$ *is a circle.*

This is a known result, but we can give its analytic proof by using Theorem 4.

4 The first variational formula

The first variational formula of $\mathcal{M}$ was calculated by [2, 3] using the integral p.v. $\iint$ in Cauchy's principal value. The absolute integrability was shown in [4] in the class $C^{3+\alpha}(\mathbb{R}/\mathcal{L}\mathbb{Z})$. Using the decomposition of Theorem 1, we can show the corresponding result on the proper domain $H^{1,\infty}(\mathbb{R}/\mathcal{L}\mathbb{Z}) \cap H^{\frac{3}{2}}(\mathbb{R}/\mathcal{L}\mathbb{Z})$.

Let $\boldsymbol{\phi}$ be a function from $\mathbb{R}/\mathcal{L}\mathbb{Z}$ to $\mathbb{R}^n$. The first variation of $\mathcal{M}_i(\boldsymbol{f})$ is defined by

$$\delta\mathcal{M}_i(\boldsymbol{f})[\boldsymbol{\phi}] = \frac{d}{d\varepsilon}\mathcal{M}_i(\boldsymbol{f} + \varepsilon\boldsymbol{\phi})\bigg|_{\varepsilon=0}.$$

Theorem 5 (First variational formula) *Assume that* $\boldsymbol{f}$ *has no self-intersection and* $\mathcal{M}(\boldsymbol{f}) < \infty$ *holds, which implies* $\boldsymbol{f} \in H^{1,\infty}(\mathbb{R}/\mathcal{L}\mathbb{Z}) \cap H^{\frac{3}{2}}(\mathbb{R}/\mathcal{L}\mathbb{Z})$ *and bi-Lipschitz. Let* G_i *be the integrand of the first variational formula of* $\mathcal{M}_1$, *i.e.,*

$$\delta\mathcal{M}_i(\boldsymbol{f})[\boldsymbol{\phi}] = \iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} G_i(s_1, s_2)ds_1ds_2 \quad (i = 1, 2),$$

where $G_i ds_1 ds_2 = \delta\mathcal{M}_i(\boldsymbol{f})ds_1ds_2 + \mathcal{M}_i(\boldsymbol{f})\{\delta(ds_1)ds_2 + ds_1\delta(ds_2)\}$.

These satisfy the following.

1. *$G_i \in L^1((\mathbb{R}/\mathcal{L}\mathbb{Z})^2)$ if $\boldsymbol{f}, \boldsymbol{\phi} \in H^{1,\infty}(\mathbb{R}/\mathcal{L}\mathbb{Z}) \cap H^{\frac{3}{2}}(\mathbb{R}/\mathcal{L}\mathbb{Z})$.*

   Furthermore there exists $C > 0$ depending on the bi-Lipschitz constant of $\boldsymbol{f}$ such that

$$|\delta\mathcal{M}_i(\boldsymbol{f})[\boldsymbol{\phi}]| \leq \|G_i\|_{L^1} \leq C\left([\boldsymbol{\tau}]^2_{\frac{1}{2},2}\|\boldsymbol{\phi}'\|_{L^\infty} + [\boldsymbol{\tau}]_{\frac{1}{2},2}[\boldsymbol{\phi}']_{\frac{1}{2},2}\right).$$

2. *$G_i \in L^\infty((\mathbb{R}/\mathcal{L}\mathbb{Z})^2)$ if $\boldsymbol{f}, \boldsymbol{\phi} \in C^{1,1}(\mathbb{R}/\mathcal{L}\mathbb{Z})$.*

3. *$G_i \in C((\mathbb{R}/\mathcal{L}\mathbb{Z})^2)$ if $\boldsymbol{f}, \boldsymbol{\phi} \in C^2(\mathbb{R}/\mathcal{L}\mathbb{Z})$.*

   where put $G_i(s,s) = (-1)^i\{\boldsymbol{\kappa}(s)\cdot\boldsymbol{\phi}''(s) - \|\boldsymbol{\kappa}(s)\|^2_{\mathbb{R}^n}\boldsymbol{\tau}(s)\cdot\boldsymbol{\phi}'(s)\}$. In particular $G_1(s,s) + G_2(s,s) = 0$.

Theorem 5 implies that $\delta\mathcal{M}_i(\boldsymbol{f})[\cdot]$ is a linear functional on $H^{1,\infty}(\mathbb{R}/\mathcal{L}\mathbb{Z}) \cap H^{\frac{3}{2}}(\mathbb{R}/\mathcal{L}\mathbb{Z})$. The formal integration by parts shows that the domain $\delta\mathcal{M}_i(\boldsymbol{f})[\cdot]$ seems to be extended to $L^2(\mathbb{R}/\mathcal{L}\mathbb{Z})$. This is true for $\delta\mathcal{M}_1(\boldsymbol{f})$ if $\boldsymbol{f}$ is in $H^3(\mathbb{R}/\mathcal{L}\mathbb{Z})$.

Theorem 6 *Assume that $\boldsymbol{f} \in H^3(\mathbb{R}/\mathcal{L}\mathbb{Z})$ and that it is bi-Lipschitz. Let $\boldsymbol{\phi} \in L^2(\mathbb{R}/\mathcal{L}\mathbb{Z})$. Then it holds that*

$$\delta\mathcal{M}_1(\boldsymbol{f})[\boldsymbol{\phi}] = \langle 2\pi(-\Delta_s)^{\frac{3}{2}}\boldsymbol{f} + \text{ l.o.t.}(\boldsymbol{f}), \boldsymbol{\phi}\rangle_{L^2},$$

$$|\langle \text{ l.o.t.}(\boldsymbol{f}), \boldsymbol{\phi}\rangle_{L^2}| \leq C(\lambda)\left(\|\boldsymbol{f}\|^4_{H^{3-\varepsilon}} + 1\right)\|\boldsymbol{\phi}\|_{L^2}.$$

A similar result for $\delta\mathcal{M}_2(\boldsymbol{f})[\cdot]$ seems to hold, which is our work in progress.

5 Future works

In [4] we have already got the second variational formula with the absolutely integrable integrand for $\boldsymbol{f} \in C^{3+\alpha}(\mathbb{R}/\mathcal{L}\mathbb{Z})$. We would like to show this fact for $\boldsymbol{f} \in H^{1,\infty}(\mathbb{R}/\mathcal{L}\mathbb{Z}) \cap H^{\frac{3}{2}}(\mathbb{R}/\mathcal{L}\mathbb{Z})$.

References

[1] S. Blatt, *Boundedness and regularizing effects of O'Hara's knot energy*, J. Knot Theory Ramifications **21** (1) (2012) 1250010, 9 pp.

[2] M. H. Freedman, Z.-X. He & Z. Wang, *Möbius energy of knots and unknots*, Ann. of Math. (2) **139** (1) (1994), 1–50.

[3] Z.-X. He, *The Euler-Lagrange equation and heat flow for the Möbius energy*, Comm. Pure Appl. Math. **53** (4) (2000), 399–431.

[4] A. Ishizeki, *Some properties of the Möbius energy and its variational formulae*, Master Thesis, Saitama University, 2013.

[5] J. O'Hara & G. Solanes, *Möbius invariant energies and average linking with circles*, arXiv:1010.3764v2, 2010.