EXISTENCE AND BLOWING UP FOR A SYSTEM OF THE DRIFT-DIFFUSION EQUATION IN R^2

Masaki Kurokiba

College of Liberal Arts, Mathematical Science Research Unit,
Muroran Institute of Technology,
Muroran Hokkaido, 050-8585, Japan

We consider a well-posedness issue of the Cauchy proglem of nonlinear parabolic elliptic system called multiple drift-diffusion equation in two space dimensions: For $i = 1, 2 \cdots m$,

$$
\text{(P)} \qquad \begin{cases} \partial_t u_i - \Delta u_i + \nabla \cdot (u_i \nabla \psi) = 0, & t > 0, x \in \mathbb{R}^2, \\ -\Delta \psi = \displaystyle\sum_{j=1}^{m} \alpha_j u_j, \quad \alpha_j = \pm 1, & t > 0, x \in \mathbb{R}^2, \\ u_i(x, 0) = u_{i0}(x), & x \in \mathbb{R}^2, \end{cases}
$$

where $u_i = u_i(t, x)$ denote the densities of i-th component and ψ denotes the potential of the pariticle field.

In [19], we have studied the system for 2 kinds of pariticle. Then it is developed by the first author that the multi-component system of drift-diffusion equation is considered in [17] and established the local well-posedenss result for the class $L^2_s(\mathbb{R}^2)$, where $s > 1$. We have shown the local and global well-posedness for the drift-diffusion equation in L^p with given initial data.

$$
\text{(P')} \qquad \begin{cases} \partial_t n - \Delta n + \kappa \nabla \cdot (n \nabla \psi) = 0, & t > 0, x \in \mathbb{R}^n, \\ \partial_t p - \Delta p - \kappa \nabla \cdot (p \nabla \psi) = 0, & t > 0, x \in \mathbb{R}^n, \\ -\Delta \psi = p - n, \quad x \in \mathbb{R}^n, \\ \qquad n(0, x) = n_0(x), \qquad p(0, x) = p_0(x). \end{cases}
$$

The above equation (P') for 2 kind of paticle in the case of $\kappa = 1$ is the typical and simplest model for the semi-conductor device simulation ([2], [11], [12], [14], [18], [20]). On the other hand, when $\kappa = -1$ and $n = 2$, the equation is strongly relevant to the model of self-interacting particles with different mass (see Biler-Nadzieja [4], see also [1], [3]) and the model of the aggregation of mold known as the Keller-Segel system [15]. It is suggested that the solution that has enough regularity (and integrability) may have the instability of the solution, namely under a certain condition on the initial data, the solution blows up in finite time. In the case of a single kind of self-interacting particles, it is also shown that the solution blow up in a finite time as one can find in [1], [3].

This property is naturally inherited to the system (P). Indeed Kurokiba-Ogawa [18] showed the finite time blow up of the solution for the two particle cases and it is generalized in [6],[7], [9], [10], [8] and [18]. On the other hand, the multi-particle system is considered by Wolansky [27] in the bounded domain case.

The argument in the Keller-Segel model systematically used that the solution is in $L^1(\mathbb{R}^2)$ ([21], [22], [23]), moreover if the solution is non-negative, L^1 norm of the solution are preserved in time. In this sence it is important to consider the existence of the solution in a function

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space included in $L^1(\mathbb{R}^2)$. To avoid some technical comlexity, we employ the weighted $L^2(\mathbb{R}^2)$ space which is slightly smoller space than $L^1(\mathbb{R}^2)$ and ensure that the L^1 norm of the solution make sense. Indeed, we introduced in the previous result [18], the weighted L^2 space defined as follows. For $s > 0$,

$$L^2_s(\mathbb{R}^2) = \{f \in L^2(\mathbb{R}^2); |x|^s f(x) \in L^2(\mathbb{R}^2)\}.$$

we first show the existence and uniqueness of the solution for the two dimensional drift-diffusion type system in a subset of $L^1(\mathbb{R}^2)$. Generalization of the local wellposedness result for the multi-component system (P) is developed by several authors. We shows the local well-posedness for the system (P) in a weighted space $L^2_s(\mathbb{R}^2)$ with $s > 1$ and shows the finite time blow up of solution. On the other hand, it is meaningful if we consider the wellposedness issue in the critical function space where the invariant scaling transform is invariant. For the solution of (P), the invariant scaling $u_\lambda(t,x) = \lambda^2 u(\lambda^2 t, \lambda x)$ and $\psi_\lambda(t,x) = u(\lambda^2 t, \lambda x)$ with $\lambda > 0$. Then $L^1(\mathbb{R}^2)$ is invariant function space under this scaling transform. While for the weighted space, $s = 1$, i.e., $L^2_1(\mathbb{R}^2)$ is the invariant space. In the former results, this critical space is eliminated since we have some difficulty to controle ψ in this case.

Theorem 1. (local well-posedness in a critical weighted space) *Let $u_{i0}(x) \in L^2_1(\mathbb{R}^2), i = 1, 2, \cdots, n$. Then there exists $T = T(\|u_{10}\|_{L^2_1}, \|u_{20}\|_{L^2_1}, \cdots, \|u_{n0}\|_{L^2_1}) > 0$ and a unique weak solution $\{u_i\}_{i=1}^n$ of* (P) *with the initial data $\{u_{i0}\}_{i=1}^n$ such that $u_i \in C([0,T]; L^2_1(\mathbb{R}^2)) \cap L^2(0,T; \dot{H}^1(\mathbb{R}^2))$. Moreover the solution has higer regularity $u_i \in C([0,T]; H^2(\mathbb{R}^2) \cap L^2_1(\mathbb{R}^2)) \cap C^1(0,T; \dot{L}^2_1(\mathbb{R}^2))$ and it is the strong solution for* (P).

The solution obtained in Theorem 1 belongs to $L^1(\mathbb{R}^2)$ and hence it follows that the solution maintain the mass conservation law and meaningful identities. First of all, if the initial data is non-negative, then the maximum principle for the parabolic equation assures that the weak solution preserves non-negative structure. This fact immediately gives the $L^1(\mathbb{R}^2)$ conservation law for the weak solution.

Proposition 2. (global existence) *Assume that u_{i0} are all non negative in $L^2_1(\mathbb{R}^2)$. If $\alpha_i = -1$ for any $i = 1, 2, \cdots n$, then the corresponding solution u obtained by Theorem 1 globally exists.*

To show the local wellposedness in the critical weighted space $L^2_1(\mathbb{R}^2)$, we need to introduce the Sobolev type inequality to controle $\nabla(-\Delta)^{-1}\psi$. In Kurokiba-Ogawa [18] it is used a variant of the Brezis-Gallouet type inequality (cf. [25]). However the critial scale $L^2_1(\mathbb{R}^2)$ was eliminated by some reason.

Moreover the subject of this study is to show the following instrability result.

Theorem 3. (finite time blow up) *We assume that $\alpha_i = 1$ for all $i = 1, 2, \cdots n$. For $s > 1$, let u_{i0} be in $L^2_s(\mathbb{R}^2)$ with $u_{i0} \geq 0$ everywhere and satisfies*

$$\sum_{i=1}^n \mu_i \int_{\mathbb{R}^2} u_{i0}\,dx > 8\pi.$$

Then the solution to (P) *blows up in a finite time. Namely, there exists $T_m < \infty$ such that*

$$\limsup_{t \to T_m} \sum_{i=1}^n \|\langle x\rangle^s u_i(t)\|_2 = \infty,$$

where $\langle x\rangle = (1 + |x|^2)^{1/2}$.

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