Stability and instability of solitary waves for nonlinear Schrödinger equations of derivative type

Masahito Ohta (Tokyo University of Science)

We study the orbital stability and instability of solitary wave solutions for nonlinear Schrödinger equations of the form

$$i\partial_t u = -\partial_x^2 u - i|u|^2\partial_x u - b|u|^4 u, \quad (t,x) \in \mathbb{R}\times\mathbb{R}, \tag{1}$$

where $b \geq 0$ is a constant. We define the energy $E : H^1(\mathbb{R}) \to \mathbb{R}$ by

$$E(v) = \frac{1}{2}\|\partial_x v\|_{L^2}^2 - \frac{1}{4}(i|v|^2\partial_x v, v)_{L^2} - \frac{b}{6}\|v\|_{L^6}^6,$$

where $(v,w)_{L^2} = \Re\int_{\mathbb{R}} v(x)\overline{w(x)}\,dx$. Then, (1) is written in a Hamiltonian form $i\partial_t u = E'(u)$ in $H^{-1}(\mathbb{R})$.

The Cauchy problem for (1) is locally well-posed in the energy space $H^1(\mathbb{R})$ (see [3]). For any $u_0 \in H^1(\mathbb{R})$, there exist $T_{\max} \in (0,\infty]$ and a unique solution $u \in C([0,T_{\max}), H^1(\mathbb{R}))$ of (1) with $u(0) = u_0$ such that either $T_{\max} = \infty$ or $T_{\max} < \infty$ and $\lim_{t\to T_{\max}} \|u(t)\|_{H^1} = \infty$. Moreover, the solution $u(t)$ satisfies

$$E(u(t)) = E(u_0), \quad Q_0(u(t)) = Q_0(u_0), \quad Q_1(u(t)) = Q_1(u_0)$$

for all $t \in [0, T_{\max})$, where Q_0 and Q_1 are defined by

$$Q_0(v) = \frac{1}{2}\|v\|_{L^2}^2, \quad Q_1(v) = \frac{1}{2}(i\partial_x v, v)_{L^2}.$$

For $\theta = (\theta_0, \theta_1) \in \mathbb{R}^2$, we define $T(\theta)v(x) = e^{i\theta_0}v(x-\theta_1)$. It is known that (1) has a two parameter family of solitary wave solutions

$$T(\omega t)\phi_\omega(x) = e^{i\omega_0 t}\phi_\omega(x-\omega_1 t),$$

where $\omega = (\omega_0,\omega_1) \in \Omega := \{(\omega_0,\omega_1)\in\mathbb{R}^2 : \omega_1^2 < 4\omega_0\}$, $\gamma = 1 + \frac{16}{3}b$,

$$\phi_\omega(x) = \tilde\phi_\omega(x)\exp\left(i\frac{\omega_1}{2}x - \frac{i}{4}\int_{-\infty}^{x} |\tilde\phi_\omega(\eta)|^2\,d\eta\right),$$

$$\tilde\phi_\omega(x) = \left\{\frac{2(4\omega_0-\omega_1^2)}{-\omega_1 + \sqrt{\omega_1^2 + \gamma(4\omega_0-\omega_1^2)}\,\cosh(\sqrt{4\omega_0-\omega_1^2}\,x)}\right\}^{1/2}.$$

We note that $\phi_\omega(x)$ is a solution of

$$-\partial_x^2\phi + \omega_0\phi + \omega_1 i\partial_x\phi - i|\phi|^2\partial_x\phi - b|\phi|^4\phi = 0, \quad x \in \mathbb{R},$$

and $\tilde{\phi}_\omega(x)$ is a solution of

$$-\partial_x^2\phi + \frac{4\omega_0 - \omega_1^2}{4}\phi + \frac{\omega_1}{2}|\phi|^2\phi - \frac{3}{16}\gamma|\phi|^4\phi = 0, \quad x \in \mathbb{R}.$$

For the case $b = 0$, Colin and Ohta [1] proved that the solitary wave solution $T(\omega t)\phi_\omega$ of (1) is stable for all $\omega \in \Omega$.

In this talk, we consider the case $b > 0$, and prove the following ([2]).

Theorem 1. *Let $b > 0$. Then there exists $\kappa = \kappa(b) \in (0,1)$ such that the solitary wave solution $T(\omega t)\phi_\omega$ of (1) is stable if $-2\sqrt{\omega_0} < \omega_1 < 2\kappa\sqrt{\omega_0}$, and unstable if $2\kappa\sqrt{\omega_0} < \omega_1 < 2\sqrt{\omega_0}$.*

Remark 1. Let $b > 0$, $\gamma = 1 + \frac{16}{3}b$, and

$$g(\xi) = \frac{2(\gamma-1)}{\xi}\tan^{-1}\frac{1+\sqrt{1+\xi^2}}{\xi}, \quad \xi \in (0,\infty).$$

Then, $g : (0,\infty) \to (0,\infty)$ is strictly decreasing and bijective. Thus, for any $b > 0$, there exists a unique $\hat{\xi} = \hat{\xi}(b) \in (0,\infty)$ such that $g(\hat{\xi}) = 1$. The constant κ in Theorem 1 is given by $\kappa = (1 + \hat{\xi}^2/\gamma)^{-1/2}$.

Remark 2. The sufficient condition $-2\sqrt{\omega_0} < \omega_1 < 2\kappa\sqrt{\omega_0}$ for stability of $T(\omega t)\phi_\omega$ is equivalent to $Q_1(\phi_\omega) > 0$, and the sufficient condition $2\kappa\sqrt{\omega_0} < \omega_1 < 2\sqrt{\omega_0}$ for instability is equivalent to $Q_1(\phi_\omega) < 0$.

References

[1] M. Colin and M. Ohta, *Stability of solitary waves for derivative nonlinear Schrödinger equation*, Ann. Inst. H. Poincaré, Anal. Non Linéaire **23** (2006), 753–764.

[2] M. Ohta, *Instability of solitary waves for nonlinear Schrödinger equations of derivative type*, preprint, arXiv:1408.5537.

[3] T. Ozawa, *On the nonlinear Schrödinger equations of derivative type*, Indiana Univ. Math. J. **45** (1996), 137–163.