Existence of solutions to complex Ginzburg-Landau type equations with subcritical nonlinearity in L^p

Daisuke Shimotsuma
(Tokyo University of Science)

Let Ω be a bounded or unbounded domain in $\mathbb{R}^N$ $(N \in \mathbb{N})$ with compact C^2 boundary, $\Omega = \mathbb{R}^N_+$ or $\Omega = \mathbb{R}^N$. In this talk we consider the following initial-boundary value problem for the complex Ginzburg-Landau type equation in L^p with $1 < p < \infty$:

$$\text{(CGL)} \qquad \begin{cases} \dfrac{\partial u}{\partial t} - (\lambda + i\alpha)\Delta u + (\kappa + i\beta)|u|^{q-1}u - \gamma u = 0 & \text{in } \Omega \times (0, \infty), \\ u = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u(\cdot, 0) = u_0 & \text{on } \Omega, \end{cases}$$

where u is a complex-valued unknown function, $i = \sqrt{-1}$, $\alpha, \beta, \gamma, \kappa \in \mathbb{R}$, $\lambda > 0$, $q \geq 1$.

The complex Ginzburg-Landau type equation with $\kappa > 0$ is studied by many authors. In particular, when $1 \leq p < \infty$, $u_0 \in L^p(\Omega)$ and $1 \leq q \leq 1 + 2p/N$, Okazawa [2] obtained that if Ω is *bounded* and $\kappa \in \mathbb{R}$ then (CGL) has a unique local C^1-in-time solution. Also, when $1 < p < \infty$, $u_0 \in L^p(\Omega)$ and $1 \leq q \leq 1 + 2p/N$, Matsumoto-Tanaka [1] proved that if $\kappa > 0$ and $|\alpha|/\lambda < 2\sqrt{p-1}/|p-2|$ then (CGL) has a unique global C^1-in-time solution.

From the above, the following questions arise:

(Q1) Does (CGL) admit a local C^1-in-time solution with $\kappa \in \mathbb{R}$ even if Ω is unbounded ?
(Q2) Does (CGL) admit a global C^1-in-time solution when $\kappa \geq 0$ and $\frac{|\alpha|}{\lambda} = \frac{2\sqrt{p-1}}{|p-2|}$?
(Q3) Does (CGL) admit a global C^1-in-time solution when $\kappa < 0$?

Concerning these questions, we obtain the following results:

Main Theorem 1 (Local Existence; cf. [3]).
Let $1 < p < \infty$, $u_0 \in L^p(\Omega)$, $1 \leq q < 1 + 2p/N$. There exist $T_0 > 0$ and a unique C^1-in-time solution to (CGL) *on* $[0, T_0]$.

Main Theorem 2 (Global Existence with $\kappa \geq 0$).
Let p, u_0, q be as in Main Theorem 1. *Let $\kappa \geq 0$ and $|\alpha|/\lambda \leq 2\sqrt{p-1}/|p-2|$. Then* (CGL) *has a unique global C^1-in-time solution.*

Main Theorem 3 (Global Existence with $\kappa < 0$).
Let p, q be as in Main Theorem 1. *Assume that $\kappa < 0$, $|\alpha|/\lambda < 2\sqrt{p-1}/|p-2|$.*
(**A**) *Let $\gamma < 0$. Then there exists $\delta_1 > 0$ such that for all $u_0 \in L^p(\Omega)$ with $\|u_0\|_{L^p} < \delta_1$,* (CGL) *has a unique global C^1-in-time solution.*
(**B**) *Assume that Ω is bounded. Then there exist $c > 0$ and $\delta_2 > 0$ such that for all $\gamma < c$ and $u_0 \in L^p(\Omega)$ with $\|u_0\|_{L^p} < \delta_2$,* (CGL) *has a unique global C^1-in-time solution.*

References

[1] T. Matsumoto and N. Tanaka, *Well-posedness for the complex Ginzburg-Landau equations*, GAKUTO Internat. Ser. Math. Sci. Appl. **32**, Gakkōtosho, Tokyo, 429–442, 2010.

[2] N. Okazawa, *Smoothing effect and strong L^2-wellposedness in the complex Ginzburg-Landau equation*, Lect. Notes Pure Appl. Math. **251**, Chapman & Hall/CRC, 265–288, 2006.

[3] D. Shimotsuma, T. Yokota and K. Yoshii, *Cauchy problem for the complex Ginzburg-Landau type equation with L^p-initial data*, Math. Bohem., to appear.

*This is a joint work with Kentarou Yoshii and Tomomi Yokota.