Blow-up in finite time for Schrödinger equations with inverse-square potentials

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In this talk we consider the following Cauchy problem for nonlinear Schrödinger equations with inverse-square potentials:

$$(\mathbf{NLS})_a \qquad \begin{cases} i\dfrac{\partial u}{\partial t} = \Big(-\Delta + \dfrac{a}{|x|^2}\Big)u + u\,K(|u|^2) & \text{in } \mathbb{R}\times\mathbb{R}^N, \\ u(0,x) = u_0(x) & \text{in } \mathbb{R}^N, \end{cases}$$

where $i = \sqrt{-1}$, $N \geq 3$, $a > -(N-2)^2/4$ and K is an integral operator defined as

$$K(f)(x) = Kf(x) := \int_{\mathbb{R}^N} k(x,y)f(y)\,dy. \tag{1}$$

If $a = 0$ and $k(x,y) = W(x-y)$ in $(\mathbf{NLS})_a$, then the equation is so-called *Hartree equation.* Ginibre-Velo [**1**] studied well and particularly showed the wellposedness of Hartree equation for the first time. On the other hand, the global solvability of $(\mathbf{NLS})_a$ for $a \neq 0$ is carried out in [**5**] under the following assumption:

(K1) k is a symmetric real-valued function, that is, $k(x,y) = k(y,x) \in \mathbb{R}$ a.a. x, $y \in \mathbb{R}^N$;

(K2) $k \in L_y^\infty(L_x^\infty) + L_y^\beta(L_x^\alpha)$ and $k - k_R \to 0$ $(R \to \infty)$ in $L_y^\beta(L_x^\alpha)$ for some α, $\beta \in [1,\infty]$ such that $\alpha \leq \beta$, $\alpha^{-1} + \beta^{-1} \leq 4/N$;

(K3) $k_- := -\min\{k,\,0\} \in L_y^\infty(L_x^\infty) + L_y^{\tilde\beta}(L_x^{\tilde\alpha})$ and $k_- - (k_-)_R \to 0$ $(R \to \infty)$ in $L_y^{\tilde\beta}(L_x^{\tilde\alpha})$ for some $\tilde\alpha$, $\tilde\beta \in [1,\infty]$ such that $\tilde\alpha \leq \tilde\beta$, $\tilde\alpha^{-1} + \tilde\beta^{-1} \leq 2/N$.

Here $L_y^\beta(L_x^\alpha)$ is the family of k such that $\|k\|_{L_x^\beta(L_y^\alpha)} = \big\|\|k\|_{L_y^\alpha}\big\|_{L_x^\beta} < \infty$ and k_R is defined as

$$k_R(x,y) := \begin{cases} k(x,y) & |k(x,y)| \leq R, \\ R & k(x,y) > R, \\ -R & k(x,y) < -R. \end{cases}$$

Note that the assumption $k - k_R \to 0$ $(R \to \infty)$ in $L_x^\beta(L_y^\alpha)$ as in **(K2)** and **(K3)** is essential when $\beta = \infty$.

If $k(x,y) = W(x-y)$ for $W \in L^\alpha(\mathbb{R}^N)$, then $k \in L_x^\infty(L_y^\alpha)$ and $\|k\|_{L_x^\infty(L_y^\alpha)} = \|W\|_{L^\alpha}$. Hence **(K1)**–**(K3)** implies that the following conditions assumed in [**1**]:

(K1)′ W is a real-valued even function, that is, $W(-x) = W(x) \in \mathbb{R}$ a.a. $x \in \mathbb{R}^N$;

(K2)′ $W \in L^\infty(\mathbb{R}^N) + L^{(N/4)\vee 1}(\mathbb{R}^N)$;

(K3)′ $W_- := -\min\{W,\,0\} \in L^\infty(\mathbb{R}^N) + L^{N/2}(\mathbb{R}^N)$.

A function u is said to be a *local weak solution* to $(\mathbf{NLS})_a$ on $I(\ni 0)$ if u belongs to $L^\infty(I;H^1(\mathbb{R}^N)) \cap W^{1,\infty}(I;H^{-1}(\mathbb{R}^N))$ and satisfies $(\mathbf{NLS})_a$ in the sense of $L^\infty(I;H^{-1}(\mathbb{R}^N))$. If I coincides with $\mathbb{R}$, then the local weak solution is said to be a *global weak solution* to $(\mathbf{NLS})_a$.

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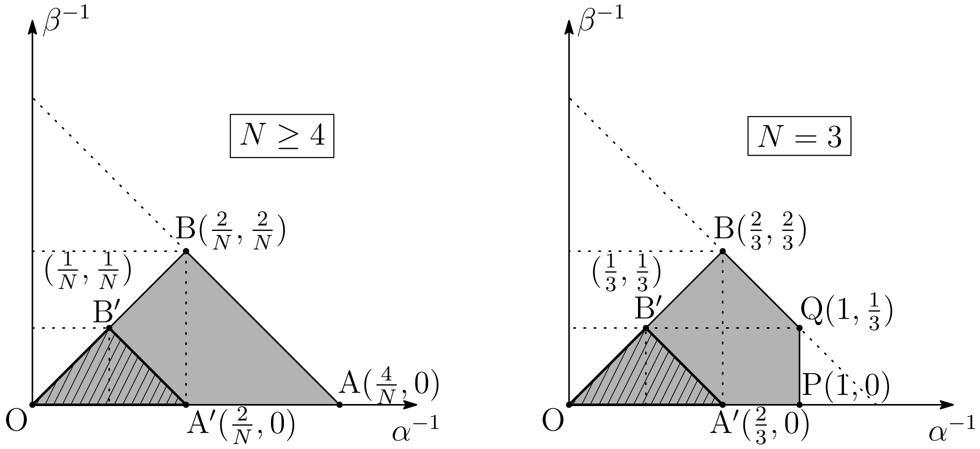

Figure 1: Admissible exponents (α, β) for **(K2)** and **(K3)**

Wellposedness([**5**]).

Let $N \geq 3$ and $a > -(N-2)^2/4$. Assume that k satisfies **(K1)**–**(K3)**. Then for every $u_0 \in H^1(\mathbb{R}^N)$ there exists a unique global weak solution u to $\mathbf{(NLS)}_a$. Moreover, u belongs to $C(\mathbb{R}; H^1(\mathbb{R}^N)) \cap C^1(\mathbb{R}; H^{-1}(\mathbb{R}^N))$ and satisfies

$$\|u(t)\|_{L^2} = \|u_0\|_{L^2}, \quad E(u(t)) = E(u_0) \quad \forall\, t \in \mathbb{R}, \tag{2}$$

where the "energy" is defined as for $\varphi \in H^1(\mathbb{R}^N)$

$$E(\varphi) := \frac{1}{2}\|\nabla\varphi\|_{L^2}^2 + \frac{a}{2}\Big\|\frac{\varphi}{|x|}\Big\|_{L^2}^2 + \frac{1}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} k(x,y)|\varphi(x)|^2|\varphi(y)|^2\,dxdy. \tag{3}$$

More precisely, there uniquely exists the local weak solution to $\mathbf{(NLS)}_a$ even when **(K3)** is not imposed.

Next we determine that the local weak solution blows up in finite time or extends globally in time when **(K3)** is not assumed. If $a = 0$, Glassey [**2**] and Ogawa-Tsutsumi [**3**] are proved blow-up in finite time. The important role of the Glassey methods is **virial identity**. The identity for $\mathbf{(NLS)}_a$ is as follows:

$$\begin{aligned}\frac{d^2}{dt^2}\|xu(t)\|_{L^2}^2 &= 16\,E(u(t)) \\ &\quad - 4\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}[k(x,y) + x\cdot(\nabla_x k)(x,y)]\,|u(t,x)|^2|u(t,y)|^2\,dxdy.\end{aligned} \tag{4}$$

To jastify (4) we need the existence of strong solutions $C(\overline{I}; H^2(\mathbb{R}^N))$ and the continuous dependence of initial values for $\mathbf{(NLS)}_a$. The continuous dependence can be shown. But strong solutions are not constructed by virtue of the singularity of the potential $|x|^{-2}$. Thus we show (4) by another approach; without construction of strong solutions to $\mathbf{(NLS)}_a$. To end this, we further assume the following condition:

(K4) $k_1 := (x\cdot\nabla_x k + y\cdot\nabla_y k)/2 \in L_x^\infty(L_y^\infty) + L_x^{\hat\beta}(L_y^{\hat\alpha})$ for some $\hat\alpha, \hat\beta \in [1,\infty]$ such that $\hat\alpha \leq \hat\beta$, $\hat\alpha^{-1} + \hat\beta^{-1} \leq 4/N$ and $k + k_1 \geq 0$.

If $k(x,y) = W(x-y)$, then **(K4)** implies that

(K4)′ $x\cdot\nabla W \in L^\infty(\mathbb{R}^N) + L^{(N/4)\vee 1}(\mathbb{R}^N)$ and $W + (1/2)x\cdot\nabla W \geq 0$.

Blow-up in finite time([**6**]).

Let $N \geq 3$ and $a > -(N-2)^2/4$. Assume that k satisfies **(K1)**, **(K2)** and **(K4)**. Then for every $u_0 \in H^1(\mathbb{R}^N)$ with $|x|u_0 \in L^2(\mathbb{R}^N)$ and $E(u_0) < 0$ the local weak solution to $\mathbf{(NLS)}_a$ blows up in finite time:

$$\lim_{t \to -T_1+0} \|\nabla u(t)\|_{L^2} = \infty = \lim_{t \to T_2-0} \|\nabla u(t)\|_{L^2}.$$

The above blow-up results generalize the well-known case for $a = 0$ and $k(x,y) = W(x-y)$. Thus we can show the blow-up in finite time for $\mathbf{(NLS)}_a$ in a way similar to the potential free case $a = 0$.

To show (4) we consider the approximate problems of $\mathbf{(NLS)}_a$:

$$\mathbf{(NLS)}_a^{\varepsilon,\delta} \qquad \begin{cases} i\dfrac{\partial u}{\partial t} = -\Delta u + \dfrac{a\,u}{|x|^2+\delta} + u\,K_\varepsilon(|u|^2) & \text{in } \mathbb{R}\times\mathbb{R}^N, \\ u(0,x) = u_0(x) & \text{in } \mathbb{R}^N, \end{cases}$$

where $\delta > 0$, $\varepsilon > 0$ and K_ε is the integral operator whose kernel is the regularization of k:

$$k_\varepsilon(x,y) = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \rho_\varepsilon(x-\xi)k(\xi,\eta)\rho_\varepsilon(y-\eta)\,d\eta\,d\xi.$$

References

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