Some examples of parametric resonance for wave equations

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An instability phenomenon which arises in a system when some parameter of the system varies in time is called parametric resonance. An example of parametric resonance is known as the existence of unbounded solutions of Hill's equation, see [1]. We first consider the Cauchy problem for the wave equation

$$\partial_t^2 u - a(t)\Delta u = 0 \quad \text{in } \mathbb{R}^{1+n} \tag{1}$$

where $a(t)$ is smooth, positive, 1-periodic and not a constant on $\mathbb{R}$. Our first result is the following:

Theorem 1 *Let u be the solution of* (1) *such that one of the initial data of u is a compactly supported smooth function, is not identically zero and the other initial data of u is identically zero. Then the energy of u grows exponentially.*

We note that Reissig and Yagdjian [3] constructed a sequence of solutions $\{u_m\}$ of (1) with smooth and compactly supported initial data such that $|u_m(m,\cdot)|$, $|\partial_{x_i} u_m(m,\cdot)|$ are uniformly greater than $C_1 \exp(C_2 m)$ in bounded domains. However the support of initial data of u_m spread out as m tends to infinity.

Next we consider

$$\begin{cases} \partial_t^2 u(t,x) - a(t)\Delta u(t,x) = 0 \quad \text{in } (0,\infty)\times\Omega, \\ u(0,x) = u_0(x), \quad \partial_t u(0,x) = u_1(x) \quad \text{in } \Omega, \\ u(t,x) = 0 \quad \text{on } (0,\infty)\times\partial\Omega \end{cases} \tag{2}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$ and $u_0 \in H^2(\Omega)\cap H_0^1(\Omega)$, $u_1 \in H_0^1(\Omega)$. Let μ be a positive eigenvalue of the Laplace operator with zero Dirichlet condition in Ω and an eigenfunction $e(x) \in H^2(\Omega)\cap H_0^1(\Omega)$. We give two examples of parametric resonance for the wave equation (2).

Theorem 2 *Let $b(t)$ be smooth, positive, 1-periodic and not a constant on $\mathbb{R}$. Then one can find a positive number λ and a one dimensional vector space V in $\mathbb{C}^2$ so that by setting $a(t) = \lambda b(t)$ the energy of the solution u of (2) grows exponentially, if the vector ${}^t\left((u_0, e)_{L^2(\Omega)} \quad (u_1, e)_{L^2(\Omega)}\right)$ is not in V.*

Theorems 1 and 2 are proved by applying the unstable properties of Hill's equation.

Theorem 3 *Let $\varphi(t) \in C^\infty([0,\infty))$ such that $\varphi(t) > 0$, $\varphi'(t) < 0$, $\varphi''(t) \geq 0$ on $[d,\infty)$ and $\varphi^{(j)}(t) \to 0$ $(t \to \infty)$ for $j = 0, 1$. Then one can find a smooth, positive and non-periodic coefficient $a(t)$ and a positive number M so that*

$$|a(t) - 1| \lesssim \varphi(\sqrt{\mu}t) - \varphi'(\sqrt{\mu}t),$$
$$|a'(t)| \lesssim \varphi(\sqrt{\mu}t) - \varphi'(\sqrt{\mu}t) + \varphi''(\sqrt{\mu}t + M)$$

on $[0,\infty)$ and for each $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ and $u_1 \in H_0^1(\Omega)$, if $(u_0, e)_{L^2(\Omega)} = 0$, $(u_1, e)_{L^2(\Omega)} \neq 0$, we have

$$\|\mathcal{R}(t,0)\,(u_0,u_1)\|_{H_0^1(\Omega)\times L^2(\Omega)} \gtrsim \exp\left(\int_0^{\sqrt{\mu}t} \varphi(s+M)\,ds\right), \quad t \gg 1,$$

and we have

$$\|\mathcal{R}(t,0)\|_{\mathrm{Hom}(H_0^1(\Omega)\times L^2(\Omega))} \lesssim \exp\left(5\int_0^{\sqrt{\mu}t} \varphi(s+M)\,ds\right) \quad \text{on } [0,\infty)$$

where $\mathcal{R}(t,0)$ is the solution operator of (2) such that $H_0^1(\Omega) \times L^2(\Omega) \ni (u_0, u_1) \mapsto (u(t,\cdot), \partial_t u(t,\cdot)) \in H_0^1(\Omega) \times L^2(\Omega)$.

We note that Colombini [2] constructed examples of parametric resonance for the Cauchy problem on one dimensional torus to the wave equation (1) with non smooth, rapidly oscillating coefficient.

References

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