ASYMPTOTIC PROFILE OF SOLUTIONS TO THE DRIFT-DIFFUSION EQUATION WITH CRITICAL DISSIPATION

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For $n \geq 2$ and $1 \leq \theta \leq 2$, we study the following initial value problem for the drift-diffusion equation:

$$
(1) \qquad \begin{cases} \partial_t u + (-\Delta)^{\theta/2} u - \nabla \cdot (u \nabla \psi) = 0, & t > 0, \ x \in \mathbb{R}^n, \\ -\Delta \psi = u, & t > 0, \ x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n, \end{cases}
$$

where $(-\Delta)^{\theta/2}\varphi = \mathcal{F}^{-1}[|\xi|^\theta \mathcal{F}[\varphi]]$ and $u_0 : \mathbb{R}^n \to \mathbb{R}_+$ is a given initial data. The drift-diffusion equation is known as the model of semiconductor device simulations. The unknown functions $u : (0,\infty) \times \mathbb{R} \to \mathbb{R}_+$ and $\nabla\psi : (0,\infty) \times \mathbb{R} \to \mathbb{R}^n$ are the density of electrons and the potential of electromagnetic field, respectively. When $\theta > 1$, (1) is parabolic. Hence the L^p theory for a parabolic equation derives well-posedness and global existence of solutions. Moreover an asymptotic expansion of the solution as $t \to \infty$ was already provided. When $\theta = 1$, the L^p theory for a parabolic equation does not work since (1) is elliptic. In the case $\theta = 1$, we introduce

$$
(2) \qquad u_\lambda(t,x) = \lambda u(\lambda t, \lambda x), \quad \psi_\lambda(t,x) = \psi(\lambda t, \lambda x).
$$

Then $(u_\lambda, \psi_\lambda)$ also fulfills the first equation on (1). The authors and Kato already proved that the solution exists globally in the Besov space which is corresponding to (2) [2]. Furthermore the solution satisfies the mass conservation $\|u(t)\|_{L^1(\mathbb{R}^n)} = \|u_0\|_{L^1(\mathbb{R}^n)}$ and

$$
(3) \qquad \|u(t)\|_{L^p(\mathbb{R}^n)} \leq C(1+t)^{-n(1-\frac{1}{p})}
$$

for $1 \leq p \leq \infty$. The decay rate on (3) is same as one of the fundamental solution of the linear equation, i.e., the Poisson kernel:

$$
P(t,x) = \pi^{-\frac{n+1}{2}} \Gamma(\tfrac{n+1}{2})\, t\, (t^2 + |x|^2)^{-\frac{n+1}{2}}.
$$

Moreover the smoothing effect of $e^{-t(-\Delta)^{1/2}}\varphi = P(t) * \varphi$ leads

$$
(4) \qquad u \in C^\infty((0,\infty); H^\infty(\mathbb{R}^n)).
$$

The large-time behavior of the solution is established in the following theorem.

Theorem 1 ([4])**.** *Let $n \geq 3$, $\theta = 1$, $(1+|x|)u_0 \in L^1(\mathbb{R}^n)$ and the solution u of* (1) *satisfy* (3) *and* (4)*. Then*

$$
\|u(t) - M_u P(t) - m_u \cdot \nabla P(t)\|_{L^q(\mathbb{R}^n)} = o\big(t^{-n(1-\frac{1}{q})-1}\big) \quad (t \to \infty)
$$

holds for $1 < q < \infty$, where $M_u = \int_{\mathbb{R}^n} u_0(y)dy$ and $m_u = \int_{\mathbb{R}^n} (-y) u_0(y) dy$.

Before considering the case $n = 2$, we refer the following corresponding problem:

$$
(5) \qquad \begin{cases} \partial_t \omega + (-\partial_x^2)^{1/2}\omega + \omega \partial_x \omega = 0, & t > 0, \ x \in \mathbb{R}, \\ \omega(0,x) = \omega_0(x), & x \in \mathbb{R}. \end{cases}
$$

For the solution ω of this Burgers equation, Iwabuchi [1] showed $\|\omega(t)-P(t)\int_{\mathbb{R}}\omega_0(y)dy\|_{L^2(\mathbb{R})}\geq c(1+t)^{-3/2}\log(2+t)$. On the other hand, the decay rate of the nonlinear term on (5) and one on (1) with $n=2$ are same. Namely $\|\omega^2(t)\|_{L^1(\mathbb{R})}=O(t^{-1})$ and $\|u\nabla\psi(t)\|_{L^1(\mathbb{R}^2)}=O(t^{-1})$ as $t\to\infty$ hold. Therefore, we can expect that the asymptotic expansion of the solution of (1) contains a logarithmic term. When $n=2$, we introduce the following function:

$$J(t,x)=\int_0^{t/2}\nabla P(t-s)*(P\nabla(-\Delta)^{-1}P)(s)ds + \int_{t/2}^{t}P(t-s)*\nabla\cdot(P\nabla(-\Delta)^{-1}P)(s)ds.$$

Since $P\nabla(-\Delta)^{-1}P$ is odd in x, $J(t)$ is well-defined on $L^1(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)$. Moreover $J\not\equiv 0$([3]). From the scaling property $\lambda^3J(\lambda t,\lambda x)=J(t,x)$ $(\lambda>0)$, we see that $\|J(t)\|_{L^p(\mathbb{R}^2)}$ has a same decay rate as $\|\nabla P(t)\|_{L^p(\mathbb{R}^2)}$. When $n=2$, the asymptotic expansion for (1) is established in the following theorem.

Theorem 2 ([4])**.** *Let $n=2,\ \theta=1,\ (1+|x|)u_0\in L^1(\mathbb{R}^2)$ and the solution u of (1) satisfy (3) and (4). Then*

$$\left\|u(t)-M_uP(t)-m_u\cdot\nabla P(t)-M_u^2J(t)\right\|_{L^q(\mathbb{R}^2)}=o\left(t^{-2(1-\frac{1}{q})-1}\right)\quad(t\to\infty)$$

holds for $1<q<\infty$, where $M_u=\int_{\mathbb{R}^n}u_0(y)dy$ and $m_u=\int_{\mathbb{R}^n}(-y)u_0(y)dy$.

Theorem 2 concludes that the large-time behavior of the solution of (1) has no logarithmic term. Theorems 1 and 2 state that decay of the solution u is decided by the total mass and the first moment of the initial data. The correction term J is derived from the following procedure. By the Duhamel formular, (1) is represented by

$$u(t)=P(t)*u_0+\int_0^t P(t-s)*\nabla\cdot(u\nabla(-\Delta)^{-1}u)(s)ds. \tag{6}$$

On the nonlinear term, we renormalize u to M_uP. Then we obtain $M_u^2J(t)$. The proof of theorems are based on the L^p-L^q estimate for $P(t)*\varphi$. For the proof, we prepare the following proposition.

Proposition 3. *Let $n\geq 2,\ \theta=1$ and the solution u of (1) satisfy (3) and (4). Then there exist positive constants C and T such that*

$$\left\|(-\Delta)^{1/4}u(t)\right\|_{L^2(\mathbb{R}^n)}\leq Ct^{-1/2}(1+t)^{-n/2} \tag{7}$$

holds for any $t\geq T$.

This proposition is shown by the energy method.

References

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