Transverse instability for a nonlinear Schrödinger equation and the stability for a bifurcation point

Yohei Yamazaki
Kyoto University

We consider the following nonlinear Schrödinger equation:

$$i\partial_t u = -\Delta u - |u|^{p-1}u, \quad (t,x,y) \in \mathbb{R} \times \mathbb{R} \times \mathbb{T}_L, \tag{1}$$

where $p > 1$ and $u = u(t,x,y)$ is an unknown complex-valued function for $t \in \mathbb{R}$, $x \in \mathbb{R}$ and $y \in \mathbb{T}_L$. Here, $\mathbb{T}_L = \mathbb{R}/2\pi L\mathbb{Z}$ and $L > 0$. Takaoka-Tzvetkov [6] showed by the Strichartz estimate that the Cauchy problem of (1) is locally well-posed in H^1. In this talk, we consider the stability for standing waves of (1).

By a standing wave, we mean a non-trivial solution of (1) with the form $u(t,x,y) = e^{i\omega t}\varphi(x,y)$, where $\omega > 0$ and $\varphi \in H^1(\mathbb{R} \times \mathbb{T}_L)$ is a solution of

$$-\Delta\varphi + \omega\varphi - |\varphi|^{p-1}\varphi = 0, \quad (x,y) \in \mathbb{R} \times \mathbb{T}_L. \tag{2}$$

Next, we define the (orbital) stability for standing waves of (1).

Definition 1. We say that the standing wave $e^{i\omega t}\varphi$ is stable in H^1 if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for all $u_0 \in H^1_{sym}(\mathbb{R} \times \mathbb{T}_L)$ with $\|u_0 - \varphi\|_{H^1} < \delta$, the solution $u(t)$ of (1) with the initial data $u(0) = u_0$ exists globally in time and satisfies

$$\sup_{t>0} \inf_{\theta\in\mathbb{R}} \left\| u(t) - e^{i\theta}\varphi \right\|_{H^1} < \varepsilon,$$

Where

$$H^s_{sym}(\mathbb{R} \times \mathbb{T}_L) := \{u \in H^s(\mathbb{R} \times \mathbb{T}_L) : u(x,y) = u(x,-y) = u(-x,y), (x,y) \in \mathbb{R} \times [-\pi L, \pi L]\}.$$

Otherwise, we say the standing wave $e^{i\omega t}\varphi$ is unstable in H^1.

Let the function φ_ω be the positive symmetric solution of

$$-\partial_x^2\varphi + \omega\varphi - |\varphi|^{p-1}\varphi = 0, \quad x \in \mathbb{R}.$$

Then, $e^{i\omega t}\varphi_\omega$ is a standing wave of the nonlinear Schrödinger equation on $\mathbb{R}$:

$$i\partial_t u = -\Delta u - |u|^{p-1}u, \quad (t,x) \in \mathbb{R} \times \mathbb{R}.$$

We define

$$e^{i\omega t}\tilde{\varphi}(x,y) := e^{i\omega t}\varphi(x), \quad (t,x,y) \in \mathbb{R} \times \mathbb{R} \times \mathbb{T}_L.$$

Then, $e^{i\omega t}\varphi_\omega$ is a standing wave of (1) and we call $e^{i\omega t}\tilde{\varphi}_\omega$ a line standing wave.

In [1], Cazenave and Lions showed that the standing wave $e^{i\omega t}\varphi_\omega$ of the nonlinear Schrödinger equation on $\mathbb{R}$ is stable for $1 < p < 5$. However, in some case a line standing wave $e^{i\omega t}\tilde{\varphi}_\omega$ is unstable for $1 < p < 5$. The following transverse instability results for the line standing wave was proved in Rousset-Tzvetkov [5] for $p = 3$ and Y. [7] for $1 < p < 5$. Let

$$L_{\omega,p} = \frac{2}{\sqrt{(p-1)(p+3)\omega}}. \tag{3}$$

Theorem 1. *Let $\omega > 0$ and $1 < p < 5$.*

(i) *If $0 < L < L_{\omega,p}$, then the line standing wave $e^{i\omega t}\tilde{\varphi}_\omega$ is stable.*
(ii) *If $L > L_{\omega,p}$, then the line standing wave $e^{i\omega t}\tilde{\varphi}_\omega$ is unstable.*

Then, we farther consider the stability for the line standing wave in the case $L = L_{\omega,p}$. We can show (i) of Theorem 1 by the Lyapunov functional method in Grillakis-Shatah-Strauss [2]. Moreover, to prove (ii) of Theorem 1, they used the linear instability for the linearized equation around the standing wave $e^{i\omega t}\tilde{\varphi}_\omega$. In the case $L = L_{\omega,p}$, since the linearized operator around $e^{i\omega t}\tilde{\varphi}_\omega$ is degenerate, we can not apply the method by [2] or the argument for the proof of Theorem 1.

In this talk, we show the following stability result by applying the bifurcation result and the stability argument in Maeda [3] and Ohta [4].

Theorem 2. *Let $\omega > 0$ and $L = L_{\omega,p}$. There exist $2 \le p_1 < p_2 < 3$ with the following properties.*

(i) *If $2 < p < p_1$, then the standing wave $e^{i\omega t}\tilde{\varphi}_\omega$ is stable.*
(ii) *If $p > p_2$, then the standing wave $e^{i\omega t}\tilde{\varphi}_\omega$ is unstable.*

References

[1] T. Cazenave and P. L. Lions, *Orbital stability of standing waves for some nonlinear Schrödinger equations*, Commun. Math. Phys. **85** (1982), 549-561.

[2] M. Grillakis, J. Shatah and W. Strauss, *Stability theory of solitary waves in the presence of symmetry. I*, J. Funct. Anal. A **74** (1987), no. 1, 160-197.

[3] M. Maeda, *Stability of bound states of Hamiltonian PDEs in the degenerate cases*, J. Funct. Anal. **263** (2012), no. 2, 511-528.

[4] M. Ohta, *Instability of bound states for abstract nonlinear Schrödinger equations*, J. Funct. Anal. **261** (2011), no. 1, 90-110.

[5] F. Rousset and N. Tzvetkov, *Transverse nonlinear instability for two-dimensional dispersive models*, Ann. I. Poincaré-AN **26** (2009) 477-496.

[6] H. Takaoka and N. Tzvetkov, *On 2D nonlinear Schrödinger equations with Data on $\mathbb{R} \times \mathbb{T}$*, J. Funct. Anal. **182** (2001) 427-442.

[7] Y. Yamazaki, *Transverse instability for a system of nonlinear Schrödinger equations*, preprint.