Non-normal form of abstract evolution equations of hyperbolic type

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Set $I := [0, T]$. Let $\{A(t);\ t \in I\}$ be a family of closed linear operators in a complex Hilbert space X and $\{B(t);\ t \in I\}$ a family of positive-definite selfadjoint operators in X. Then we consider the abstract Cauchy problem for linear evolution equations of the form

$$\text{(ACP)} \qquad \begin{cases} B(t)(d/dt)u(t) + A(t)u(t) = f(t), & t \in I, \\ u(0) = u_0. \end{cases}$$

Here the initial value u_0 is selected as in Theorem 2 (see below).

First we introduce our assumption on $\{B(t);\ t \in I\}$.

Assumption on $\{B(t)\}$. The family $\{B(t)\}$ satisfies the following three conditions:

(**B1**) For $t \in I$, $B(t)$ is positive-definite selfadjoint in X with

$$\|w\|_{X_t} := \|B(t)w\| \geq \|w\| \quad \text{for } w \in X_t := D(B(t)),\ t \in I.$$

(**B2**) For $t \in I$, $X_0 = X_t$ and $B(\cdot) \in C_*(I; L(X_0, X))$, where the subscript $_*$ is used to refer the strong operator topology in $L(X_0, X)$. (for this notation see Kato [**3**]).

(**B3**) There exists a nonnegative function $\gamma \in L^1(I)$ such that

$$\big|\|w\|_{X_t} - \|w\|_{X_s}\big| \leq \Big|\int_s^t \gamma(r)\, dr\Big| \|w\|_{X_s}, \quad w \in X_0,\ t, s \in I.$$

To introduce our assumption on $\{A(t);\ t \in I\}$ we need one more family $\{S(t);\ t \in I\}$ of auxiliary operators in X.

Assumption on $\{S(t)\}$. Let $\{S(t)\}$ be a family of closed linear operators satisfying the following four conditions:

(**S1**) For $t \in I$, $B(t)S(t)$ is also positive-definite selfadjoint in X with

$$\mathrm{Re}\,(B_n(t)u, S(t)u) \geq \|B_n(t)u\|^2, \quad u \in D(S(t)),\ t \in I,\ n \in \mathbb{N},$$

where $B_n(t) := B(t)(1 + n^{-1}B(t))^{-1}$ $(n \in \mathbb{N})$ is the Yosida approximation to $B(t)$.

Then $Y_t := D((B(t)S(t))^{1/2})$ forms a Hilbert space with norm

$$\|u\|_{Y_t} := \|(B(t)S(t))^{1/2}u\|.$$

It should be noted that conditions (**B1**) and (**S1**) yield that

$$D(B(t)S(t)) \subset D(S(t)) \subset Y_t = D((B(t)S(t))^{1/2}) \subset X_t = D(B(t))$$

and

$$(B(t)u, S(t)u) \geq \|B(t)u\|^2, \quad u \in D(S(t)) \subset D(B(t)),\ t \in I.$$

(**S2**) Let $t \in I$ and $g \in X$. Then for any $\varepsilon \in (0, 1]$ there exists $u_\varepsilon(t) \in D(S(t))$ such that

$$B(t)u_\varepsilon(t) + \varepsilon S(t)u_\varepsilon(t) = g. \tag{1}$$

It follows from conditions (**B1**), (**S1**) and (**S2**) that $\{B(t)+\varepsilon S(t);\ \varepsilon \in (0, 1]\}$ is a family of **semi-Fredholm operators** in X (see Kato [**2**, Section IV.5]). Namely, $B(t) + \varepsilon S(t)$ is closed and invertible with closed range. Thus the index of $B(t) + \varepsilon S(t)$ is constant for $\varepsilon \in (0, 1]$. Therefore it suffices to assume that (1) holds for some $\varepsilon_0 \in (0, 1]$.

(**S3**) For $t \in I$, $Y_0 = Y_t$ and $(B(\cdot)S(\cdot))^{1/2} \in C_*(I; L(Y_0, X))$.

(**S4**) There exists a nonnegative function $\sigma \in L^1(I)$ such that $\sigma \geq \gamma$ and

$$\big|\|v\|_{Y_t} - \|v\|_{Y_s}\big| \leq \Big|\int_s^t \sigma(r)\, dr\Big| \|v\|_{Y_s}, \quad v \in Y_0,\ t, s \in I.$$

Remark 1. The condition (**B3**) and (**S4**) imply that $\|w\|_t$ and $\|u\|_{Y_t}$ are (uniformly) absolute continuous in $t \in I$, respectively. These conditions are equivalent to the following conditions

(**B3**)′ There exists a nonnegative function $\gamma' \in L^1(I)$ such that

$$\|w\|_{X_t} \leq \exp\Big(\Big|\int_s^t \gamma'(r)\, dr\Big|\Big) \|w\|_{X_s}, \quad w \in X_0,\ t, s \in I.$$

(**S4**)′ There exists a nonnegative function $\sigma' \in L^1(I)$ such that $\sigma' \geq \gamma'$ and

$$\|v\|_{Y_t} \leq \exp\Big(\Big|\int_s^t \sigma'(r)\, dr\Big|\Big) \|v\|_{Y_s}, \quad v \in Y_0,\ t, s \in I.$$

This type of expression are found in Kato [**3**].

Let $\{B(t)\}$ and $\{S(t)\}$ be as defined above. Then we may introduce the following

Assumption on $\{A(t)\}$. The family $\{A(t)\}$ satisfies the following four conditions:

(**A1**) $Y_t \subset D(A(t)) \subset X_t$, $t \in I$.

(**A2**) There exists a constant $\alpha \geq 0$ such that

$$|\mathrm{Re}\,(A(t)v, B(t)v)| \leq \alpha \|B(t)v\|^2, \quad v \in D(A(t)),\ t \in I.$$

(**A3**) There exists a constant $\beta \geq \alpha$ such that for $u \in D(S(t)) \subset D((B(t)S(t))^{1/2})$,

$$|\mathrm{Re}\,(A(t)u, S(t)u)| \leq \beta(B(t)u, S(t)u) = \beta\|(B(t)S(t))^{1/2}u\|^2, \quad t \in I.$$

(**A4**) $A(\cdot) \in C_*(I; L(Y_0, X))$.

If $\{A(t)\}$ satisfies conditions (**A1**)–(**A4**), then $\{-A(t)\}$ also does. This is the reason why we employ the term, hyperbolic type, when we refer our evolution equations.

Theorem 1. *Suppose that* Assumptions on $\{B(t)\}$, $\{A(t)\}$ *and* $\{S(t)\}$ *are satisfied. Then there exists a unique evolution operator* $\{U(t, s); (t, s) \in \Delta_+\}$ *for* (ACP), *where* $\Delta_+ := \{(t, s);\ 0 \leq s \leq t \leq T\}$, *having the following properties*:

(**i**) $U(t,s)X_0 \subset X_0$ *and* $U(\cdot,\cdot)$ *is strongly continuous on* Δ_+ *to* $L(X_0)$, *with*

$$\|U(t,s)\|_{L(X_s,X_t)} \le \exp\Big(\int_s^t \widetilde{\alpha}(r)\,dr\Big), \quad (t,s)\in\Delta_+,$$
$$\|U(t,s)\|_{L(X_0)} \le \exp\Big(2\int_0^s \gamma(r)\,dr\Big)\exp\Big(\int_s^t \big(\widetilde{\alpha}(r)+\gamma(r)\big)\,dr\Big), \quad (t,s)\in\Delta_+,$$

where $\widetilde{\alpha}(r) := \alpha + \gamma(r)$.

(**ii**) $U(t,r)U(r,s) = U(t,s)$ *on* Δ_+ *and* $U(s,s) = 1$ (*the identity on* X_0).

(**iii**) $U(t,s)Y_0 \subset Y_0$ *and* $U(\cdot,\cdot)$ *is strongly continuous on* Δ_+ *to* $L(Y_0)$, *with*

$$\|U(t,s)\|_{L(Y_s,Y_t)} \le \exp\Big(\int_s^t \widetilde{\beta}(r)\,dr\Big), \quad (t,s)\in\Delta_+,$$
$$\|U(t,s)\|_{L(Y_0)} \le \exp\Big(2\int_0^s \sigma(r)\,dr\Big)\exp\Big(\int_s^t \big(\widetilde{\beta}(r)+\sigma(r)\big)\,dr\Big), \quad (t,s)\in\Delta_+,$$

where $\widetilde{\beta}(r) := \beta + \sigma(r)$.

Furthermore, let $v \in Y_0$. *Then* $U(\cdot\,,\cdot)v \in C^1(\Delta_+; X_0)$, *with*

(**iv**) $B(t)(\partial/\partial t)U(t,s)v = -A(t)U(t,s)v, \quad (t,s)\in\Delta_+$, *and*

(**v**) $(\partial/\partial s)U(t,s)v = U(t,s)B(s)^{-1}A(s)v, \quad (t,s)\in\Delta_+$.

The equation in (ACP) is naturally interpreted if the solution has an additional property $u(\cdot)\in C(I;Y_0)$. In fact, it is guaranteed by condition (**A1**) that $u(t)\in Y_0 \subset D(A(t))$ for every $t\in I$.

Theorem 2. *Let* $\{U(t,s)\}$ *be the evolution operator for* (ACP) *as in* Theorem 1 *above. For* $u_0 \in Y_0$ *and* $f(\cdot)\in C(I;X)$ *satisfying* $B(\cdot)^{-1}f(\cdot)\in L^1(I;Y_0)$, *define* $u(\cdot)$ *as*

$$u(t) := U(t,0)u_0 + \int_0^t U(t,s)B(s)^{-1}f(s)\,ds.$$

Then (ACP) *has a unique* (*classical*) *solution*

$$u(\cdot)\in C^1(I;X_0)\cap C(I;Y_0).$$

Remark 2. Theorems 1 and 2 are nothing but generalizations of the corresponding theorems in [**4**].

References

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[4] N. Okazawa and K. Yoshii, *Linear Schrödinger evolution equations with moving Coulomb singularities*, J. Differential Equations **254** (2013), 2964–2999.