# Dynamical behavior of integral equations in epidemiology

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Due to various environmental changes such as diversity of transport network and a rapid increase in population, disease tranmission has been considered as a serious threat to global population despite medical development. In view of mathematical biology, the transmission models have played a crucial role to predict and theoretically understand the eventual disease prevalence in the host population. In terms of Volterra-type integral equations, structured population models have widely been formulated to investigate the effect of **differently aged individuals** for infectious disease process (see, e.g., Diekmann et al. [1]). One of the structured models is formulated by **age of infection**, denoting the time that has elapsed since the infection has started.

We consider the asymptotic behavior of the following epidemiological model:

$$\begin{cases} \dfrac{dS(t)}{dt} = B - S(t)\displaystyle\int_0^{+\infty} \beta(a)b(t-a)e^{-(\mu+\eta+\gamma)a}\,da - \mu S(t) + \delta R(t), \\ b(t) = S(t)\displaystyle\int_0^{+\infty} \beta(a)b(t-a)e^{-(\mu+\eta+\gamma)a}\,da, \\ \dfrac{dR(t)}{dt} = \gamma\displaystyle\int_0^{+\infty} b(t-a)e^{-(\mu+\eta+\gamma)a}\,da - (\mu+\delta)R(t) \end{cases} \qquad (*)$$

with the initial conditions $(S(0), b(\theta), R(0)) = (s, \phi(\theta), r)$ for $\theta \leq 0$. Let $\mathbb{R}_+$ (resp. $\mathbb{R}_-$) denote the set of nonnegative (resp. nonpositive) real numbers. We assume that $(s, \phi, r) \in \mathbb{R}_+ \times L^1_\rho(\mathbb{R}_-; \mathbb{R}_+) \times \mathbb{R}_+$, the space consists of all equivalence classes of measurable functions $\phi : \mathbb{R}_- \longrightarrow \mathbb{R}_+$ such that $\|\phi\|_{L^1_\rho} = \int_0^{+\infty} e^{-\rho a}|\phi(-a)|da < +\infty$. The meaning of the variables and coefficients is as follows.

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| $S(t)$ : | number of susceptible individuals at time $t$ |
| $b(t)$ : | an incidence rate (number of newly infected individuals) at time $t$ |
| $R(t)$ : | number of recovered individuals at time $t$ |
| $\mathcal{F}(a)$ : | a survival rate to be infected until his or her infection-age becomes $a$ with $\mathcal{F}(0) = 1$ |
| $\beta(a)$ : | an age-specific transmission coefficient of infected individual at age $a$ |
| $B$ : | a birth rate of susceptible individuals |
| $\mu$ : | a natural mortality rate |
| $\eta$ : | a disease-induced death rate |
| $\gamma$ : | a recovery rate |
| $\delta$ : | a rate of immunity loss of recovered individuals |

Under the condition that $\beta \in L^\infty(\mathbb{R}_+)$, the basic reproduction number $R_0$ (**the average number of secondary infections** by a single infective individual), given as the dominant eigenvalue of a positive linear operator, is calculated as

$$R_0 = B\int_0^{+\infty} \beta(a)\mathcal{F}(a)\,da.$$

We first establish asymptotic staility of an endemic equilibrium of the model $(*)$ and offer a several open problems when the rate of immunity loss $\delta$ is positive for the case $R_0 > 1$ [2, Section 3]. Furthermore, the reformulation method from the model $(*)$ into a system of DDEs (delay differential equations) is introduced when $\beta$ is a function of bounded variation on $\mathbb{R}_+$ [2, Section 4].

# References

[1] O. Diekmann, M. Gyllenberg, H. Huang, M. Kirkilionis, J.A.J. Metz and H.R. Thieme, On the formulation and analysis of general deterministic structured population models: II. Nonlinear theory, J. Math. Biol. **43** (2001) 157-189.

[2] Y. Nakata, Y. Enatsu, H. Inaba, T. Kuniya, Y. Muroya and Y. Takeuchi, Stability of epidemic models with waning immunity, SUT Journal of Mathematics **50** (2014) 405-426.