Classification of complex valued solutions for a nonlinear Klein-Gordon equation and its extension by the momentum

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1 Introduction

In this talk, we consider the following nonlinear Klein-Gordon equation:

$$\text{(NLKG)} \qquad \begin{cases} \partial_t^2 u - \Delta u + u = |u|^2 u, & (t,x) \in \mathbb{R} \times \mathbb{R}^3, \\ (u(0), \partial_t u(0)) = (u_0, u_1), & x \in \mathbb{R}^3, \end{cases}$$

where u is a complex valued unknown function and $(u(0), \partial_t u(0))$ belongs to $H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3) =: \mathcal{H}$. It is well known that this equation (NLKG) is locally well-posed in the energy space $\mathcal{H}$. Moreover, the energy, the charge, and the momentum are conserved by the flow. Here these are defined by

$$\text{(Energy)} \qquad E(u(t), \partial_t u(t)) := \frac{1}{2}\|u(t)\|_{H^1}^2 + \frac{1}{2}\|\partial_t u(t)\|_{L^2}^2 - \frac{1}{4}\|u(t)\|_{L^4}^4,$$

$$\text{(Charge)} \qquad C(u(t), \partial_t u(t)) := \mathrm{Im} \int_{\mathbb{R}^3} \overline{u(t)} \partial_t u(t) dx,$$

$$\text{(Momentum)} \qquad P(u(t), \partial_t u(t)) := \mathrm{Re} \int_{\mathbb{R}^3} \nabla u(t) \overline{\partial_t u(t)} dx,$$

where $\mathrm{Im} f$ denotes the imaginary part of f and $\mathrm{Re} f$ denotes the real part of f. We would like to know the global behavior of the solutions to (NLKG), for example, scattering, blow-up and so on.

2 Main Results

For $\omega \in [0,1)$, we call $e^{i\omega t} Q_\omega$ standing wave if Q_ω is the minimum energy solution to

$$-\Delta Q_\omega + (1-\omega^2) Q_\omega - |Q_\omega|^2 Q_\omega = 0, \quad x \in \mathbb{R}^3.$$

We note that the standing waves are the global but non-scattering solutions to (NLKG). Payne and Sattinger [5] classified the real valued solutions whose energy are less than that of the ground state Q_0 to global and blow-up. Ibrahim, Masmoudi, and Nakanishi [2] proved that the global real valued solution below the ground state Q_0 scatters using the method of Kenig and Merle [3]. We will extend this result to complex valued solutions below the standing wave $e^{i\omega t} Q_\omega$. More precisely, we prove the following theorem.

Theorem 2.1. *Fix $\omega \in [0,1)$. Let (u_0, u_1) be complex valued functions in $\mathcal{H}$ and u denote the solution of (NLKG) with $(u(0), \partial_t u(0)) = (u_0, u_1)$. We define the two subsets $\mathcal{K}_\omega^{\pm}$ by*

$$\mathcal{K}_\omega^{+} := \{(u_0,u_1) \in \mathcal{H}; EC_\omega(u_0,u_1) < m_\omega, K(u_0) \geq 0\},$$
$$\mathcal{K}_\omega^{-} := \{(u_0,u_1) \in \mathcal{H}; EC_\omega(u_0,u_1) < m_\omega, K(u_0) < 0\},$$

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where $EC_\omega(u_0,u_1) := E(u_0,u_1) - \omega|C(u_0,u_1)|$ *and* $K := 2\|\nabla u_0\|_{L^2}^2 - 3\|u_0\|_{L^4}^4/2$. *Then the following are valid:*

(i) *If* $(u_0,u_1) \in \mathcal{K}_\omega^+$, *then the solution* $u(t)$ *of (NLKG) scatters in both time directions, that is,* u *exists globally and there exist solutions* $v_\pm$ *of the free Klein-Gordon equation such that*

$$\|(u(t), \partial_t u(t)) - (v_\pm(t), \partial_t v_\pm(t))\|_{\mathcal{H}} \to 0 \text{ as } t \to \pm\infty.$$

(ii) *If* $(u_0,u_1) \in \mathcal{K}_\omega^-$, *then the solution* $u(t)$ *of (NLKG) blows up in finite time in both time directions, that is, there exist times* $T_+, T_- \in (0,\infty)$ *such that the maximal existense time interval is* $(-T_-, T_+)$ *and* $\lim_{t\to\pm T_\pm \mp 0} \|(u(t), \partial_t u(t))\|_{\mathcal{H}} = \infty$.

This theorem coinsides with the result by Ibrahim, Masmoudi, and Nakanishi [2] if $\omega = 0$. Due to this theorem, we could classify the solutions which have the larger energy than that of the ground state Q_0. To prove (i), we use the argument based on [3, 2]. On the other hand, we cannot use their proof, which is based on Payne and Sattinger [5], to obtain the blow-up result (ii). So we combine the idea of Holmer and Roudenko [1] with the idea of Ohta and Todorova [4].

Moreover, we consider the following extension by the momentum in this talk.

Theorem 2.2. *Let* $\omega \in [0,1)$ *and* $v \in \mathbb{R}^3$ *with* $|v| < 1$. *We define the functional* $K_{1,0}^{v,\omega}$ *and the subsets* $\mathcal{K}_{v,\omega}^\pm$ *in* $\mathcal{H}$ *respectively as follows:*

$$K_{1,0}^{v,\omega}(u,s) := \|\nabla u\|_{L^2}^2 + \{1 - \omega^2(1-|v|^2)\}\|u\|_{L^2}^2 - \|u\|_{L^4}^4 - \|v\cdot\nabla u\|_{L^2}^2 + 2s\omega\sqrt{1-|v|^2}\,\mathrm{Im}\int_{\mathbb{R}^3} v\cdot\nabla u\bar{u}dx$$

where $s = \pm 1$, *and*

$$\mathcal{K}_{v,\omega}^+ := \left\{(u_0,u_1) \in \mathcal{H}; EPC_{v,\omega}(u_0,u_1) < \sqrt{1-|v|^2}m_\omega,\ K_{1,0}^{v,\omega}(u_0, \mathrm{sign}(C(u_0,u_1))) \geq 0\right\},$$
$$\mathcal{K}_{v,\omega}^- := \left\{(u_0,u_1) \in \mathcal{H}; EPC_{v,\omega}(u_0,u_1) < \sqrt{1-|v|^2}m_\omega,\ K_{1,0}^{v,\omega}(u_0, \mathrm{sign}(C(u_0,u_1))) < 0\right\},$$

where $EPC_{v,\omega}(u_0,u_1) := E(u_0,u_1) + v\cdot P(u_0,u_1) - \omega\sqrt{1-|v|^2}|C(u_0,u_1)|$. *Then the follwing two statements hold:*

(i) *If* $(u_0,u_1) \in \mathcal{K}_{v,\omega}^+$, *then the solution* $u(t)$ *of (NLKG) scatters.*

(ii) *If* $(u_0,u_1) \in \mathcal{K}_{v,\omega}^-$, *then the solution* $u(t)$ *of (NLKG) blows up in finite time.*

The proof of Theorem 2.2 is based on the Lorentz transform. $K_{1,0}^{v,\omega}$ is generated from $K_{1,0}^\omega(u) := \|\nabla u\|_{L^2}^2 + (1-\omega^2)\|u\|_{L^2}^2 - \|u\|_{L^4}^4$ by the Lorentz transform. We note that we can replace K by $K_{1,0}^\omega$ in Theorem 2.1 since $K_{1,0}^\omega$ is the same sign as K if $EC_\omega < m_\omega$.

References

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