# Existence of blow–up solutions for a discrete semilinear heat equation

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In this talk we consider the following partial difference equation:

$$f_{\vec{n}}^{t+1} = \frac{g_{\vec{n}}^t}{\{1-\alpha\delta(g_{\vec{n}}^t)^\alpha\}^{1/\alpha}}, \tag{dSH}$$

where $t \in \mathbb{Z}_{\geq 0},\ \vec{n} \in \mathbb{Z}^d,\ \alpha, \delta > 0$,

$$g_{\vec{n}}^t := \sum_{k=1}^{d} \frac{f_{\vec{n}+\vec{e}_k}^t + f_{\vec{n}-\vec{e}_k}^t}{2d},$$

and $\vec{e}_k \in \mathbb{Z}^d$ is a unit vector whose $k$–th component is 1. From (dSH), we obtain

$$\frac{f_{\vec{n}}^{t+1} - f_{\vec{n}}^t}{\delta} = \sum_{k=1}^{d} \frac{f_{\vec{n}+\vec{e}_k}^t - 2f_{\vec{n}}^t + f_{\vec{n}-\vec{e}_k}^t}{2d\delta} + (g_{\vec{n}}^t)^{1+\alpha} + O(\delta)\ (\delta \to 0).$$

Let $\xi := \sqrt{2d\delta},\ T = \delta t,\ \vec{X} = \xi\vec{n}$. Taking a limit: $\delta \to +0$, we obtain the following semilinear heat equation:

$$\frac{\partial f}{\partial T} = \Delta f + f^{1+\alpha}, \tag{SH}$$

where $f := f(T, \vec{X}),\ T \geq 0,\ \vec{X} \in \mathbb{R}^d$ and $\Delta$ is a $d$–dimensional Laplacian. From this property, we call (dSH) discrete semilinear heat equation in this talk. There are solutions for (SH) which are called blow–up solution as follow.

**Definition 1** *If there exists positive number $T_b$ such that*

$$\limsup_{T \to T_b - 0} \|f(T, \cdot)\|_{L^\infty} = +\infty,$$

*where $\|f(T,\cdot)\|_{L^\infty} := \sup_{\vec{X} \in \mathbb{R}^d} |f(T, \vec{X})|$, we call $f$ which is a solution for* (SH) *blow–up solution and $T_b$ blow–up time.*

Moreover, Fujita, Weissler et. al. proved the following theorem [1–4].

**Theorem 1** *Consider Cauchy problem of* (SH)*:*

$$\begin{cases} \dfrac{\partial f}{\partial T} = \Delta f + f^{1+\alpha}, \\ f(0, \vec{X}) = a(\vec{X}). \end{cases}$$

*If the initial condition $a(\vec{X})$ is smooth and satisfies $a(\vec{X}) \geq 0, \not\equiv 0$, following statements hold.*

1. *If $0 < \alpha \leq 2/d$, there exists a finite time $T_b > 0$ and the solution for* (SH) *blows up at time $T_b$.*

2. *If $2/d < \alpha$, the solution for* (SH) *does not blow up at any finite time for sufficiently small initial value $a(\vec{X})$.*

In contrast, since we obtain

$$f_{\vec{n}}^{s+1} = \frac{g_{\vec{n}}^{s}}{\{1 - \alpha\delta(g_{\vec{n}}^{s})^{\alpha}\}^{1/\alpha}} \xrightarrow{g_{\vec{n}}^{s} \to (\alpha\delta)^{-1/\alpha} - 0} +\infty$$

from (dSH), there are similar solutions for (dSH) to blow–up solutions for (SH). We also call such solutions defined as follow blow–up solution.

**Definition 2** *If $t < t_0 \in \mathbb{Z}_{>0}$, $g_{\vec{n}}^{t} < (\alpha\delta)^{-1/\alpha}$ is satisfied for all $\vec{n} \in \mathbb{Z}^d$ and if $t = t_0$, there exists $\vec{n}_0 \in \mathbb{Z}^d$ and $g_{\vec{n}_0}^{t_0} \geq (\alpha\delta)^{-1/\alpha}$ is satisfied, i.e. $1 - \alpha\delta(g_{\vec{n}_0}^{t_0})^{\alpha} \leq 0$, then we say the solution for* (dSH) *blows up at time $t_0$.*

Furthermore, similar theorem to that of (SH) holds [5]. The following theorem is main theorem of this talk.

**Theorem 2** *Consider Cauchy problem of* (dSH)*:*

$$\begin{cases} f_{\vec{n}}^{t+1} = \dfrac{g_{\vec{n}}^{t}}{\{1 - \alpha\delta(g_{\vec{n}}^{t})^{\alpha}\}^{1/\alpha}}, \\ f_{\vec{n}}^{0} = a_{\vec{n}}. \end{cases}$$

*If the initial condition $a_{\vec{n}}$ satisfies $a_{\vec{n}} \geq 0, \not\equiv 0$, following statements hold.*

1. *If $0 < \alpha \leq 2/d$, there exists a finite time $t_b > 0$ and the solution for* (dSH) *blows up at time $t_b$.*

2. *If $2/d < \alpha$, the solution for* (dSH) *does not blow up at any finite time for sufficiently small initial value $a_{\vec{n}}$.*

# References


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