Scale invariant elliptic operators with singular coefficients*

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1. Introduction

In this talk we deal with generation of analytic semigroups on $L^p = L^p(\mathbb{R}^N)$ by elliptic operators of the form

$$L = |x|^\alpha \Delta + c|x|^{\alpha-2} x \cdot \nabla - b|x|^{\alpha-2}, \quad x \in \mathbb{R}^N \setminus \{0\},$$

where $c, b \in \mathbb{R}$ and $\alpha \in \mathbb{R}$ with $\alpha \neq 2$ are fixed constants. In the case $N = 1$, we consider L in $\mathbb{R}_+ = (0, \infty)$ on $L^p(\mathbb{R}_+)$.

For the case $\alpha = c = 0$, L forms Schrödinger operators with inverse-square potentials which has been studied in many papers. In [**3**] Okazawa gave the necessary and sufficient condition on generation of analytic **contraction** semigroups on L^p by L_{min} (the closure of L defined on $C_0^\infty(\mathbb{R}^N \setminus \{0\})$, see also Notations below).

The purpose of this talk is to give the necessary and sufficient condition on generation of analytic, **bounded** and positive semigroups on $L^p(\mathbb{R}^N)$ by suitable realizations of L with $\alpha, c, b \in \mathbb{R}$ and $1 < p < \infty$.

Notations

- L_{min} : the closure of L defined on $C_0^\infty(\mathbb{R}^N \setminus \{0\})$,
- L_{max} : L in L^p in the sense of distributions on $\mathbb{R}^N \setminus \{0\}$.

2. Result

Before stating our result, we introduce a quadratic polynomial which is closely related to Euler's indicial equation for $|x|^{2-\alpha} L[u(r)] = 0$ $(r = |x|)$:

$$f(s) = b + s(N - 2 + c - s), \quad s \in \mathbb{R}$$

and we denote its roots by s_1, s_2 $(s_1 \le s_2)$ when $D_c := b + \left(\frac{N-2+c}{2}\right)^2 \ge 0$, that is,

$$s_1 = \frac{N-2+c}{2} - \sqrt{b + \left(\frac{N-2+c}{2}\right)^2}, \quad s_2 = \frac{N-2+c}{2} + \sqrt{b + \left(\frac{N-2+c}{2}\right)^2}.$$

In fact $|x|^{-s_1}$ and $|x|^{-s_2}$ are both radial solutions to $Lu = 0$.

*This is a joint work with Professors Giorgio Metafune and Chiara Spina (University of Salento) and Professor Noboru Okazawa (Tokyo University of Science).

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Now we state our result on analytic semigroups generated by L.

Theorem 1 ([**2**]). *Let $1 < p < \infty$, $\alpha \neq 2$, $D_c = b + (\frac{N-2+c}{2})^2 > 0$. Then a suitable realization L_{int} of L in L^p (satisfying $L_{min} \subset L_{int} \subset L_{max}$) generates an analytic C_0-semigroup on L^p if and only if*

$$s_1 + \min\{0, 2-\alpha\} < \frac{N}{p} < s_2 + \max\{0, 2-\alpha\}.$$

In this case the generated analytic C_0-semigroup is bounded and positive. The domain of L_{int} is given by

$$D(L_{int}) = \{u \in D(L_{max})\ ;\ |x|^{\theta(\alpha-2)}u \in L^p\},$$

where $\theta \in (0, 1]$ satisfies

$$s_1 + \min\{0, 2-\alpha\} < \frac{N}{p} + \theta(\alpha - 2) < s_2 + \max\{0, 2-\alpha\}.$$

Theorem 2 ([**2**]). *Let $1 < p < \infty$, $\alpha \neq 2$, $D_c = b + (\frac{N-2+c}{2})^2 = 0$ and $s_0 = \frac{N-2+c}{2}$. Then a suitable realization L_{int} of L in L^p (satisfying $L_{min} \subset L_{int} \subset L_{max}$) generates an analytic C_0-semigroup in L^p if and only if*

$$s_0 + \min\{0, 2-\alpha\} < \frac{N}{p} < s_0 + \max\{0, 2-\alpha\}.$$

In this case the generated analytic C_0-semigroup is bounded and positive. The domain of L_{int} is given by

$$D(L_{int}) = \begin{cases} \left\{u \in D(L_{max})\ ;\ \chi_{\{|x|<1/2\}}|x|^{\theta_0(\alpha-2)}\big|\log|x|\big|^{-2/p}u \in L^p\right\} & \text{if } \alpha < 2, \\ \left\{u \in D(L_{max})\ ;\ \chi_{\{|x|>1/2\}}|x|^{\theta_0(\alpha-2)}\big|\log|x|\big|^{-2/p}u \in L^p\right\} & \text{if } \alpha > 2, \end{cases}$$

where $\theta_0 \in (0, 1)$ satisfies

$$s_0 = \frac{N}{p} + \theta_0(\alpha - 2).$$

Remark 1. In the case $\alpha = 2$, L_{min} generates an analytic semigroup on L^p and coincides with L_{max} for every $c, b \in \mathbb{R}$ and $1 < p < \infty$. In addition, the generated analytic semigroup by L is bounded in a sector if and only if $s_1 < N/p < s_2$.

References

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