$(-\Delta+|x_1|^a+|x_2|^b)u=0$ の 正値解の構造について

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目標：$0\le a\le b$ に対しシュレディンガー方程式

(1) $Lu\equiv(-\Delta+|x_1|^a+|x_2|^b)u=0$ on $D=\mathbb{R}^2$

の正値解の構造を決定したい. 特に、ここでは L に関する D のマルチン境界とマルチン核 ([1]) について、これまで解明された部分を報告する.

記号.

$\partial_M D$: Martin boundary of D for L.

$\partial_m D$: minimal Martin boundary of D for L.

$D^*=D\cup\partial_M D$: Martin compactification of D for L.

$K(x,\xi)$ $(x\in D,\xi\in\partial_M D)$: Martin kernel for L.

Fact (cf.[2]) Let $a=b=0$ or 2. Then

$$\partial_M D=\partial_m D=\infty S^1,\quad D^*=\mathbb{R}^2\cup\infty S^1,$$

where ∞S^1 is the sphere at infinity. For $\omega\in S^1$,

$$K(x,\infty\omega)=\begin{cases}e^{\sqrt{2}x\cdot\omega} & a=b=0\\ e^{-|x|^2/2}\psi(x\cdot\omega)/\psi(0) & a=b=2\end{cases},$$

where $x_0=0$ and

$$\psi(z)=\int_0^\infty e^{-s^2+2zs}ds,\quad z\in\mathbb{R}.$$

Suppose $b>0$ in the following.

$L_1=-d^2/dx_1^2+|x_1|^a$ on $D_1=\mathbb{R}$.

$L_2=-d^2/dx_2^2+|x_2|^b$ on $D_2=\mathbb{R}$.

$p_k(w,z,t)$: min. fundamental sol. for ∂_t+L_k on $D_k\times(0,\infty)$ $(k=1,2)$.

$\partial_M D_2$: Martin boundary of D_2 for L_2.

$\partial_m D_2$: min. Martin boundary of D_2 for L_2.

$D_2^*=D_2\cup\partial_M D_2$: Martin comp. of D_2 for L_2.

$\lambda_0<\lambda_1\le\lambda_2\le\cdots$: eigenvalues of L_2

ϕ_0: normalized pos. eigenfunction for λ_0,

H_0: pos. Green function of $L_1+\lambda_0$ on D_1.

$\partial_M D_1$: Martin boundary of D_1 for $L_1+\lambda_0$.

$\partial_m D_1$: min. Martin boundary of D_1 for $L_1+\lambda_0$.

$D_1^*=D_1\cup\partial_M D_1$: Martin comp. of D_1 for $L_1+\lambda_0$.

K_0: Martin kernel for $L_1+\lambda_0$, i.e.,

$$K_0(x_1,\xi_1)=\lim_{D_1\ni y_1\to\xi_1}\frac{H_0(x_1,y_1)}{H_0(x_1^0,y_1)},\quad x_1\in D_1,\ \xi_1\in\partial_M D_1,$$

It is known that for $k=1,2$,

$$D_k^*=[-\infty,\infty],\quad \partial_m D_k=\partial_M D_k=\{-\infty,\infty\}.$$

Theorem 1. *Let $0\le a\le 2<b$.*

(i) With d_2 being an ideal point standing for degeneracy of D_2^* to one point,

$$D^*=\partial_M D_1\times\{d_2\}\cup D_1\times D_2^*.$$

Here a subset $U_\pm$ of D^* is a neighborhood of $(\pm\infty,d_2)$ if and only if

$$U_\pm\supset\{(\pm\infty,d_2)\}\cup\{x_1\in\mathbb{R};N<\pm x_1<\infty\}\times D_2^*$$

for some $N>1$. Furthermore,

$$\partial_m D=\partial_M D=\partial_M D_1\times\{d_2\}\cup D_1\times\partial_M D_2.$$

(ii) For $\xi=(\xi_1,d_2)$ with $\xi_1\in\partial_M D_1$,

$$K(x,\xi)=K_0(x_1,\xi_1)\phi_0(x_2)/\phi_0(x_2^0),\quad x\in D.$$

(iii) For $\xi=(\xi_1,\xi_2)\in D_1\times\partial_M D_2$,

(2) $$K(x,\xi)=H(x,\xi)/H(x^0,\xi),\quad x\in D,$$

(3) $$H(x,\xi)=\int_0^\infty p_1(x_1,\xi_1,t)q(x_2,\xi_2,t)dt.$$

Here $q(x_2,\xi_2,t)$ is a continuous function on $D_2\times\partial_M D_2\times\mathbb{R}$ defined by

$$q(x_2,\xi_2,t)=0\quad\text{for}\quad t\le 0,$$
$$q(x_2,\xi_2,t)=\lim_{D_2\ni y_2\to\xi_2}q(x_2,y_2,t)\quad \textit{for}\quad t>0,$$

where $q(x_2,y_2,t)=p_2(x_2,y_2,t)/\phi_0(y_2)$.

Theorem 2. *Let $a>2$ and $b>2$.*

(i) $D^*=D_1^*\times D_2^*$. Furthermore,

$$\partial_m D=\partial_M D=\partial_M D_1\times D_2^*\cup D_1\times\partial_M D_2.$$

(ii) For $\xi \in \partial_M D_1 \times D_2^*$,

$$K(x,\xi) = k(x,\xi)/k(x^0,\xi), \quad x \in D,$$
$$k(x,\xi) = \int_0^\infty r(x_1,\xi_1,t) q(x_2,\xi_2,t)dt.$$

Here $r(x_1,\xi_1,t)$ is a continuous function on $D_1 \times \partial_M D_1 \times \mathbb{R}$ defined by

$$r(x_1,\xi_1,t) = 0 \quad \text{for} \quad t \le 0,$$
$$r(x_1,\xi_1,t) = \lim_{D_1 \ni y_1 \to \xi_1} r(x_1,y_1,t) \quad \textit{for} \quad t > 0,$$

where $r(x_1,y_1,t) = p_1(x_1,y_1,t)/H_0(x_1^0,y_1)$.

(iii) For $\xi \in D_1 \times \partial_M D_2$, $K(x,\xi)$ is determined by (2) and (3).

For $0 < b \le 2$, put

$$I_b(z) = \begin{cases} \max(\log|z|,0) & \text{for} \quad b=2 \\ (1-b/2)^{-1}|z|^{1-b/2} & \text{for} \quad 0<b<2 \end{cases}.$$

Set $\theta_0 = \sqrt{\lambda_0+1}$. For $-\theta_0 < \theta < \theta_0$, put

$$C_{\theta,\pm} = \{(y_1,y_2) \in \mathbb{R}^2; y_1 = \theta I_b(y_2), \pm y_2 \in [2,\infty)\} \cup \{\beta_{\theta,\pm\infty}\},$$

where $\beta_{\theta,\pm\infty}$ is an ideal point standing for

$$\lim_{y_2 \to \pm\infty} (\theta I_b(y_2), y_2).$$

Set

$$B_\pm = \bigcup_{-\theta_0<\theta<\theta_0} C_{\theta,\pm},$$
$$\partial^\infty B_\pm = \{\beta_{\theta,\pm\infty}; -\theta_0 < \theta < \theta_0\}.$$

Put

$$A_\pm = \{(y_1,y_2) \in \mathbb{R}^2;\ \pm y_1 \ge \theta_0 \max(I_b(y_2), I_b(2))\} \cup \{\alpha_{\pm\infty}\},$$

where $\alpha_{\pm\infty}$ is an ideal point standing for degeneracy of vertical segments

$$A_\pm \cap \{y_1 = \pm N\}$$

to one point as $N \to \infty$. Set

$$\partial^\infty A_\pm = \{\alpha_{\pm\infty}\},$$
$$R = D \setminus (A_+ \cup B_+ \cup A_- \cup B_-)$$
$$= \{(y_1,y_2) \in \mathbb{R}^2; |y_1| < \theta_0 I_b(2), |y_2| < 2\}.$$

Theorem 3. *Let $a = 0 < b \le 2$. Then*

$$D^* = A_+ \cup B_+ \cup A_- \cup B_- \cup R.$$

Here $B_\pm$ is equipped with the standard topology, and a fundamental neighborhood system of $\alpha_{\pm\infty}$ is given by the family $\{F_{\pm,\rho,N}\}_{0<\rho<\theta_0<N}$, where

$$F_{\pm,\rho,N} = \{\alpha_{\pm\infty}\} \cup \{\beta_{\theta,\infty}; \rho < \pm\theta < \theta_0\}$$
$$\cup \{\beta_{\theta,-\infty}; \rho < \pm\theta < \theta_0\}$$
$$\cup \{(y_1,y_2) \in \mathbb{R}^2; \pm y_1 > N, |y_1| > \rho I_b(y_2)\}.$$

Furthermore,

$$\partial_m D = \partial_M D$$
$$= \partial^\infty A_+ \cup \partial^\infty B_+ \cup \partial^\infty A_- \cup \partial^\infty B_-,$$
$$K(x,\alpha_{\pm\infty}) = e^{\pm\theta_0(x_1-x_1^0)} \phi_0(x_2)/\phi_0(x_2^0),$$
$$K(x,\beta_{\theta,\pm\infty}) = e^{\theta(x_1-x_1^0)} M_{L_2+1-\theta^2}(x_2,\pm\infty),$$

where $M_{L_2-\lambda}(x_2,\pm\infty)$ is the Martin kernel for $L_2 - \lambda$ on $D_2 = \mathbb{R}$ with reference point x_2^0. In particular, if $b = 2$, for $-1 < \lambda < \lambda_0 = 1$

$$M_{L_2-\lambda}(x_2,\pm\infty) = \psi_\lambda^\pm(x_2)/\psi_\lambda^\pm(x_2^0),$$
$$\psi_\lambda^\pm(z) = e^{-z^2/2} \int_0^\infty s^{-(\lambda+1)/2} e^{-s^2 \pm 2zs} ds.$$

Key of the proof.

- $b > 2 \Leftrightarrow p_2$ is intrinsically ultracontractive (IU): $\exists$ pos. decreasing function $C(t)$ on $(0,1]$ s.t.

$$p_2(z,w,t) \le C(t)\phi_0(z)\phi_0(w),$$
$$z,w \in D_2, \quad 0 < t \le 1.$$

- $$a \le (>) 2 \Rightarrow \lim_{y_1 \to \xi_1} \frac{H_1(x_1,y_1)}{H_0(x_1^0,y_1)} = (>) 0.$$

Note. In [3] they needed the additional condition $\lim_{t\downarrow 0} t \log C(t) = 0$.

参考文献

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[2] M. Murata, Duke Math. J. **53** (1986), 869–943.

[3] M. Murata and N. Suzuki, Potential Anal. **40** (2014), 279–305.