GLOBAL WELL-POSEDNESS ON THE DERIVATIVE NONLINEAR SCHRÖDINGER EQUATION

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Abstract: We consider the Cauchy problem of the nonlinear Schrödinger equation with derivative,

$$\begin{cases} i\partial_t u + \partial_x^2 u = i\partial_x(|u|^2 u), & t \in \mathbb{R}, x \in \mathbb{R}, \\ u(0,x) = u_0(x) \in H^1(\mathbb{R}). \end{cases} \tag{0.1}$$

The equation in (0.1) is L^2-critical and completely integrable.

Local well-posedness for the Cauchy problem (0.1) is well-understood. It was proved in the energy space $H^1(\mathbb{R})$ by Hayashi and Ozawa(1993,1994), and earlier by Tsutsumi and Fukuda (1980,1981), Guo and Tan (1991) in smooth spaces. In this talk, we consider the global well-posedness.

The H^1-solution of (0.1) obeys the following mass, energy, and momentum conservation laws,

$$\begin{aligned} M(u(t)) &:= \int_{\mathbb{R}} |u(t,x)|^2\,dx = M(u_0), \\ E_D(u(t)) &:= \int_{\mathbb{R}} \Big(|u_x(t,x)|^2 + \frac{3}{2}\mathrm{Im}|u(t,x)|^2 u(t,x)\overline{u_x(t,x)} + \frac{1}{2}|u(t,x)|^6 \Big)\,dx = E_D(u_0), \\ P_D(u(t)) &:= \mathrm{Im}\int_{\mathbb{R}} \overline{u(t,x)}u_x(t,x)\,dx - \frac{1}{2}\int_{\mathbb{R}} |u(t,x)|^4\,dx = P_D(u_0). \end{aligned}$$

Using mass and energy conservation laws, and the gauge transformations, Hayashi and Ozawa (1994,1996) proved that (0.1) is globally well-posed in the energy space $H^1(\mathbb{R})$ under the condition

$$\|u_0\|_{L^2} < \|Q\|_{L^2}.$$

Here Q is the unique (up to some symmetries) positive solution of the elliptic equation,

$$-Q_{xx} + Q - \frac{3}{16}Q^5 = 0.$$

In this talk, we use the three conservation laws to prove the following two theorems. The proofs are simple, based on the different arguments.

Theorem 0.1. *There exists a small $\varepsilon_* > 0$ such that for any $u_0 \in H^1(\mathbb{R})$ with*

$$\int_{\mathbb{R}} |u_0(x)|^2\,dx < \int_{\mathbb{R}} |Q(x)|^2\,dx + \varepsilon_*,$$

the Cauchy problem (0.1) ($\lambda = 1$) is globally well-posed in $H^1(\mathbb{R})$ and the solution u satisfies

$$\|u\|_{L_t^\infty H_x^1} \leqslant C(\varepsilon_*, \|u_0\|_{H^1}).$$

The result implies that for (0.1), the ground state mass $\int_{\mathbb{R}} |Q(x)|^2\,dx$ is not the threshold of the global well-posedness and blow-up.

Let W be the unique (up to some symmetries) positive solution of the elliptic equation,

$$-W_{xx} + W^3 - \frac{3}{16}W^5 = 0.$$

Then $\int_{\mathbb{R}} |W(x)|^2\,dx = 2\int_{\mathbb{R}} |Q(x)|^2\,dx$. The second theorem is an improvement of Theorem 0.1.

Theorem 0.2. *For any* $u_0 \in H^1(\mathbb{R})$ *with*

$$\int_{\mathbb{R}} |u_0(x)|^2\,dx < \int_{\mathbb{R}} |W(x)|^2\,dx,$$

the Cauchy problem (0.1) *is globally well-posed in* $H^1(\mathbb{R})$ *and the solution* u *satisfies*

$$\|u\|_{L_t^\infty H_x^1} \leqslant C(\|u_0\|_{H^1}).$$