On the asymptotic growth rate of the sum of p-adic-valued i.i.d.

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1 Introduction

For a fixed prime integer p, the p-adic norm of an integer z is defined by $|z|_p = p^{-m}$ if $z \neq 0$ and m is the maximum integer such that p^m divides z, and by $|0|_p = 0$. The p-adic norm is extended to rational numbers in a natural manner ;

$$\left|\frac{z}{z'}\right|_p = \frac{|z|_p}{|z'|_p},\ z, z' \in \mathbf{Z}.$$

The p-adic field $\mathbf{Q}_p$ is the completion of $\mathbf{Q}$ relative to the topology induced by the p-adic norm, and identified with the set of formal p-power series ;

$$\mathbf{Q}_p = \left\{ \sum_{i=-m}^{\infty} a_i p^i \mid a_i \in \{0, 1, \dots, p-1\}, a_{-m} \neq 0 \right\} \cup \{0\},$$

with p-adic norm given by $\left|\sum_{i=-m}^{\infty} a_i p^i\right|_p = p^m$ if $a_{-m} \neq 0$. The addition and multiplication rules are defined similarly as those of polynomials.

If we fix a non-trivial character φ_1 of $\mathbf{Q}_p$, we can identify the additive group $\mathbf{Q}_p$ with its character group $\mathbf{Q}_p^*$ by the isomorphism

$$\mathbf{Q}_p \ni y \mapsto \varphi_y(\cdot) := \varphi_1(y\ \cdot\ ) \in \mathbf{Q}_p^*.$$

Then we can consider the characteristic function of a probability measure μ on $\mathbf{Q}_p$,

$$\hat{\mu}(y) = \int_{\mathbf{Q}_p} \varphi_y(x)\mu(dx),\ y \in \mathbf{Q}_p,$$

as a function on $\mathbf{Q}_p$.

2 Preceding results

We shall review some limit theorems concerning infinitely divisible distributions, semi-selfdecomposable distributions, and semistable laws on the p-adic field (some are given for more general spaces).

To start with, let X be a locally compact second countable abelian group.

Definition 1. A distribution μ on X is said to be **infinitely divisible**, if for every natural number n there exist a distribution μ_n on X and an element x_n of X such that $\mu = \mu_n^{*n} * \delta_{x_n}$.

Definition 2. A distribution ν on X is said to be **idempotent**, if $\nu = \nu * \nu$.

Proposition 1 ([6])**.** *A distribution μ on X is infinitely divisible and has no non-degenerate idempotent factor, if and only if it is the weak limit of the law of the sum $x_0 + \sum_{i=1}^{k(n)} \xi_{ni}$, for some $x_0 \in X$ and some double sequence of random variables ξ_{ni} $(n \geq 1, 1 \leq i \leq k(n))$ on X which are independent in each row such that*

- *ξ_{ni} are infinitesimal,*
- $\sup_n \sum_{i=1}^{k(n)} \left(1 - \left\| \hat{\mathcal{L}}_{\xi_{ni}}(\varphi) \right\|\right) < \infty$ *for any character φ of X,*

where $\mathcal{L}_{\xi_{ni}}$ is the law of ξ_{ni}.

In case X has a field structure, we can consider semi-selfdecomposable distributions as a subclass of infinitely divisible distributions. Suppose X is a locally compact second countable topological field.

Definition 3. Let $a \in X$ be such that $\lim_{n\to\infty} a^n = 0$. A distribution μ on X is said to be ***a*-semi-selfdecomposable**, if there exists an infinitely divisible distribution ν on X with no non-degenerate idempotent factor, such that $\hat{\mu}(\varphi) = \hat{\mu}(\varphi_a)\hat{\nu}(\varphi)$ for any character φ of X, where $\varphi_a(\cdot) := \varphi(a\ \cdot\ )$.

Proposition 2 ([6])**.** *A distribution μ on X is a-semi-selfdecomposable if and only if it is the weak limit of the law of the scaled sum $x_0 + a_n \sum_{i=1}^{k(n)} \xi_i$, for some $x_0 \in X$, some sequence of independent random variables ξ_i, a non-decreasing sequence $k(n)$ of natural numbers, a sequence $a_n \in X$ with $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = a$, such that*

- *$a_n \xi_i$ are infinitesimal,*

- $\sup_n \sum_{i=1}^{k(n)} \left(1 - \left\|\hat{\mathcal{L}}_{a_n\xi_i}(\varphi)\right\|\right) < \infty$ *for any character* φ *of* X.

Let us confine ourselves to the case $X = \mathbf{Q}_p$, and deal with a further subclass of distributions, the semistable laws.

Definition 4. For $\alpha > 0$, a distribution μ on $\mathbf{Q}_p$ whose characteristic function is given by $\hat{\mu}(y) = \exp\left(-c|y|_p^\alpha\right)$ for some constant $c > 0$ is called an α**-semistable law**.

We can see semistable laws are characterized as weak limits of scaled sums of i.i.d. Let ξ_i, $i = 1, 2, \dots$, be p-adic valued i.i.d. whose laws are rotation-symmetric, $T(p^n) = P(|\xi_i|_p \geq p^n)$, $n \in \mathbf{Z}$, be their tail probabilities, and denote $S_n = \sum_{i=1}^n \xi_i$ for $n \geq 1$.

Proposition 3 ([6])**.** *A distribution* μ *on* $\mathbf{Q}_p$ *is semistable if and only if it is the weak limit of the law of the scaled sum* $p^n S_{k(n)}$ *of some rotation-symmetric i.i.d.* ξ_i *and a non-decreasing sequence* $k(n)$ *of natural numbers satisfying* $\sup_n k(n)T\left(p^{n+l}\right) < \infty$ *for any* $l \in \mathbf{Z}$.

Proposition 4 ([6])**.** *If the p-adic valued rotation-symmetric i.i.d.* ξ_i *have tail probabilities of the form* $T(p^n) = p^{-\alpha n}L(n)$ *for some* $\alpha > 0$ *and a function* $L(n) > 0$ *of* $n \in \mathbf{Z}$ *satisfying* $\lim_{n\to\infty} \frac{L(n+1)}{L(n)} = 1$*, then the law of the scaled sum* $p^n S_{[N(n)]}$ *weakly converges to an* α*-semistable law, where* $N(n) = CT(p^n)^{-1}$ *with any constant* C.

3 Main results

Let us consider p-adic valued rotation-symmetric i.i.d. ξ_i, $i = 1, 2, \dots$, subject to the conditions in Proposition 4. We can see the convergence of the scaled sum $p^n S_{[N(n)]}$ in Proposition 4 cannot be concluded in any stronger sense, such as the convergence in probability. In fact the scaled sum converges weakly with infinitely large amplitude.

Proposition 5. *We have*

$$\liminf_{n\to\infty} \left|p^n S_{[N(n)]}\right|_p = 0, \quad \limsup_{n\to\infty} \left|p^n S_{[N(n)]}\right|_p = +\infty, \; a.s.$$

Our aim is to find the true growth rate of $|S_{[N(n)]}|_p$, namely the critical scaling order of the sum $S_{[N(n)]}$ over (resp. under) which the lim sup of the norm of the scaled sum is 0 (resp. $+\infty$) almost surely.

Theorem 1. *Let* $\beta > 0$ *and* $c_n = [\beta \log n]$.

(i). If $\beta > \frac{1}{\alpha \log p}$*, then* $\limsup_{n\to\infty} \left|p^{n+c_n} S_{[N(n)]}\right|_p = 0$*, a.s.*

(ii). If $\beta < \frac{1}{\alpha \log p}$*, then* $\limsup_{n\to\infty} \left|p^{n+c_n} S_{[N(n)]}\right|_p = +\infty$*, a.s.*

In this theorem nothing is persented for the case $\beta = \frac{1}{\alpha \log p}$. We can see that, at the critical order $c_n = \left[\frac{\log n}{\alpha \log p}\right]$ the behavior of the scaled sum depends more sensitively to the asymptotics of the tail probabilities $T(p^n) = p^{-\alpha n} L(n)$ of the i.i.d. More precisely, the asymptotics of the scaled sum differs by the rate of the convergence $\frac{L(n+1)}{L(n)} \to 1$.

Theorem 2. *Let* $c_n = \left[\frac{\log n}{\alpha \log p}\right]$*, and suppose* $\frac{L(n+1)}{L(n)} = (\log n)^{-\gamma/\log n}$, $\gamma > 0$.

(i). If $\gamma > \alpha \log p$*, then* $\limsup_{n\to\infty} \left|p^{n+c_n} S_{[N(n)]}\right|_p = 0$*, a.s.*

(ii). If $\gamma \leq \alpha \log p$*, then* $\limsup_{n\to\infty} \left|p^{n+c_n} S_{[N(n)]}\right|_p = +\infty$*, a.s.*

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