Nonlocal solutions of hyperbolic type equations*

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1 Abstract result

Set $I := [0, T]$. Let $\{A(t); t \in I\}$ be a family of closed linear operators in a complex Hilbert space X. Then we consider existence and uniqueness of (classical) solutions to nonlocal Cauchy problems for nonlinear evolution equations of the form

$$\text{(P)} \qquad \begin{cases} (d/dt)u(t) + A(t)u(t) = f(t) + \Gamma(t, K(t)u)g(t), \quad t \in I, \\ u(0) = u_0 + Mu. \end{cases}$$

By [**2**, Theorem 1.2], introducing an auxiliary family of operators $\{S(t); t \in I\}$, we can show that $A(t)$ has a unique evolution operator $\{U(t,s); (t,s) \in I \times I\}$. Thus we also introduce the following assumptions:

Assumption on $\{S(t)\}$. The family $\{S(t); t \in I\}$ satisfies the following conditions:

(**S1**) For every $t \in I$, $S(t)$ is positive selfadjoint in X and

$$(u, S(t)u) \geq \|u\|^2 \quad \text{for } u \in D(S(t)).$$

Let $Y_t := D(S(t)^{1/2})$ be the Hilbert space with $(u,v)_{Y_t} := (S(t)^{1/2}u, S(t)^{1/2}v)$, $\|u\|_{Y_t} := (u,u)_{Y_t}^{1/2}$ for $t \in I$ and $u, v \in Y_t$. In particular, we set $Y := Y_0$.

(**S2**) For $t \in I$, $Y_t = Y$, and $S(\cdot)^{1/2} \in C_*(I; L(Y, X))$.

(**S3**) $\exists \sigma \in L^1(I)$; for $(t,s) \in \Delta_+ := \{(t,s); 0 \leq s \leq t \leq T\}$,

$$\exp\Big(-\int_s^t \sigma(r)\,dr\Big)\|S(s)^{1/2}v\| \leq \|S(t)^{1/2}v\| \leq \exp\Big(\int_s^t \sigma(r)\,dr\Big)\|S(s)^{1/2}v\|, \quad v \in Y.$$

Assumption on $\{A(t)\}$. The family $\{A(t)\}$ satisfies the following four conditions:

(**A1**) $\exists \alpha \geq 0$; $|\operatorname{Re}(A(t)v, v)| \leq \alpha\|v\|^2,\ v \in D(A(t)),\ t \in I.$

(**A2**) $Y \subset D(A(t))$, $t \in I$.

(**A3**) $\exists \beta \geq \alpha$; $|\operatorname{Re}(A(t)u, S(t)u)| \leq \beta\|S(t)^{1/2}u\|^2,\ u \in D(S(t)) \subset Y,\ t \in I.$

(**A4**) $A(\cdot) \in C_*(I; L(Y, X))$.

Assumption on Γ, $\{K(t)\}$ and M. $\Gamma \in C(I \times \mathbb{C})$, $K(\cdot) \in C_*(I; L(C(I;Y), \mathbb{C}))$, $g(\cdot) \in C(I;X) \cap L^1(I;Y)$, $M \in L(C(I;Y);Y)$ and

(**ΓM**) $\|M\|_{L(C(I;Y);Y)} + \liminf_{n\to\infty} \dfrac{C_n}{n}\|g\|_{L^1(I;Y)} < e^{-\beta T - 2\|\sigma\|_{L^1(I)}}$, where

$$k_0 := \sup_{t\in I}\|K(t)\|_{L(C(I;Y),\mathbb{C})}, \quad C_n := \max\{|\Gamma(t,h)|; t \in I, |h| \leq nk_0\} \quad \text{for } n \in \mathbb{N}.$$

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Definition 1 (Nonlocal (classical) solution to (P))**.** A vector-valued function $u : I \to X$ is said to be a nonlocal solution to (P) if $u \in C^1(I;X) \cap C(I;Y)$ and satisfies (P).

Theorem 1 ([**1**])**.** *Suppose that* Assumptions on $\{A(t)\}$, $\{S(t)\}$, Γ, $\{K(t)\}$ *and* M *are satisfied. Then for* $u_0 \in Y$ *and* $f(\cdot) \in C(I;X) \cap L^1(I;Y)$, *Problem* (P) *has a nonlocal* (*classical*) *solution*

$$u(\cdot) \in C^1(I;X) \cap C(I;Y).$$

In particular, if we add the condition;

$(\mathbf{\Gamma M})'$ $\|M\|_{L(C(I;Y);Y)} + \limsup_{n\to\infty} \frac{C_n}{n}\|g\|_{L^1(I;Y)} < e^{-\beta T - 2\|\sigma\|_{L^1(I)}}$,

(**Lip**) $\exists L > 0$; $|\Gamma(t,h_1) - \Gamma(t,h_2)| \le L|h_1 - h_2|$ $(t \in I, h_1, h_2 \in B(0,R))$ *and*

$$\|M\|_{L(C(I;Y);Y)} + Lk_0\|g\|_{L^1(I;Y)} < e^{-\beta T - 2\|\sigma\|_{L^1(I)}},$$

where $R > 0$ *given by* $(\mathbf{\Gamma M})'$. *Then* $u(\cdot)$ *is a unique solution.*

2 Application to Schrödinger equation

Theorem 1 can be applied to the nonlocal Cauchy problem for the nonlinear Schrödinger equation:

$$\text{(NLS)} \quad \begin{cases} i\dfrac{\partial}{\partial t}u(x,t) - \Delta u(x,t) + V(x,t)u(x,t) = f(x,t) \\ \qquad\qquad + \gamma\Big(t, \dfrac{1}{t}\displaystyle\int_0^t \Big(\int_{\mathbb{R}^3} a(y,s)\Delta u(y,s)\,dy\Big)ds\Big)g(x,t), & (x,t) \in \mathbb{R}^3 \times I, \\ u(x,0) = u_0(x) + \displaystyle\int_0^T b(s)u(x,s)\,ds, & x \in \mathbb{R}^3. \end{cases}$$

Define $\Sigma^2(\mathbb{R}^3) := \{u \in H^2(\mathbb{R}^3); (1+|x|^2)u \in L^2(\mathbb{R}^3)\}$. Then we obtain the following

Theorem 2 ([**1**])**.** *Let* $V \in W^{1,1}(I; (L^2 + \langle x\rangle^2 L^\infty)(\mathbb{R}^3;\mathbb{R}))$, *where*

$$\langle x\rangle^2 L^\infty(\mathbb{R}^3) := \{f \in L^\infty_{\mathrm{loc}}(\mathbb{R}^3); (1+|x|^2)^{-1}f \in L^\infty(\mathbb{R}^3)\}.$$

Assume that $g \in C(I;L^2(\mathbb{R}^3)) \cap L^1(I;\Sigma^2(\mathbb{R}^3))$, $a \in C(I;L^2(\mathbb{R}^3))$, $\gamma \in C(I \times \mathbb{C})$, $b \in L^1(I)$ *and there exists a constant* $L \ge 0$ *satisfying*

$$|\gamma(t,h_1) - \gamma(t,h_2)| \le L|h_1 - h_2|, \quad t \in I,\ h_1, h_2 \in \mathbb{C},$$
$$\|b\|_{L^1(I)} + L\|a\|_{C(I;L^2)}\|g\|_{L^1(I;\Sigma^2)} < e^{-\beta T - 2\|\sigma\|_{L^1(I)}},$$

where $\beta \in \mathbb{R}$ *and* $\sigma \in L^1(I)$ *depend on only* V. *Then for every initial value* $u_0 \in \Sigma^2(\mathbb{R}^3)$ *and* $f \in C(I;L^2(\mathbb{R}^3)) \cap L^1(I;\Sigma^2(\mathbb{R}^3))$, *Problem* (NLS) *has a unique* (*classical*) *solution*

$$u(\cdot) \in C^1(I;L^2(\mathbb{R}^3)) \cap C(I;\Sigma^2(\mathbb{R}^3)).$$

References

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