Classification of bifurcation diagrams for supercritical elliptic equations in a ball

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We study the global bifurcation diagram of the solutions of the supercritical semilinear elliptic Dirichlet problem

$$\begin{cases} \Delta u + \lambda f(u) = 0 & \text{in } \ B, \\ u > 0 & \text{in } B, \\ u = 0 & \text{on } \partial B, \end{cases} \tag{1}$$

where $B = \{x \in \mathbf{R}^N : |x| < 1\}$ with $N \geq 3$, $\lambda \geq 0$ is a parameter,

$$f \in C^1([0,\infty)) \quad \text{and} \quad f(u) > 0 \quad \text{for } 0 \leq u < \infty,$$

and $f(u)$ has the form

$$f(u) = u^p + g(u) \tag{2}$$

with $p > p_S := (N+2)/(N-2)$. In (2) we assume that $g(u)$ satisfies

$$|g(u)| \leq C_0 u^{p-\delta} \quad \text{and} \quad |g'(u)| \leq C_0 u^{p-1-\delta} \quad \text{for } u \geq u_0 \tag{3}$$

with some positive constants u_0, δ and C_0.

By the symmetry result of Gidas-Ni-Nirenberg [5], every regular positive solution u is radially symmetric and $\|u\|_{L^\infty} = u(0)$. Let $\mathcal{C}$ denote the set of all the regular solution of (1). It is known that the solution set $\mathcal{C}$ can be parametrized by $\alpha := \|u\|_{L^\infty}$ (see, e.g., [9]), that is, $\mathcal{C}$ becomes a curve and is described as

$$\mathcal{C} = \{(\lambda(\alpha), u(r,\alpha)) : u(0,\alpha) = \alpha,\ 0 < \alpha < \infty\}.$$

There are several results about bifurcation diagrams of supercritical elliptic equations. Joseph-Lundgren [7] studied the Dirichlet problem

$$\begin{cases} \Delta u + \lambda (u+1)^p = 0 & \text{in } \ B, \\ u > 0 & \text{in } \ B, \\ u = 0 & \text{on } \ \partial B. \end{cases} \tag{4}$$

Define the exponent p_{JL} by

$$p_{JL} := \begin{cases} 1 + \dfrac{4}{N - 4 - 2\sqrt{N-1}}, & N \geq 11, \\ \infty, & 2 \leq N \leq 10, \end{cases}$$

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which is called the Joseph-Lundgren exponent introduced in [7]. It was shown by [7] that there exists $\lambda^* > 0$ and the following holds: When $p_S < p < p_{JL}$, $\lambda(\alpha)$ oscillates infinitely many times around λ^* and converges to λ^* as $\alpha \to \infty$, and when $p \geq p_{JL}$, $\lambda(\alpha)$ is strictly increasing and converges to λ^* as $\alpha \to \infty$.

The study of the problem

$$\begin{cases} \Delta u + \lambda u + u^p = 0 & \text{in } B, \\ u > 0 & \text{in } B, \\ u = 0 & \text{on } \partial B \end{cases} \tag{5}$$

was initiated by Brezis-Nirenberg [1] in the critical case $p = p_S$. Later, the supercritical case $p > p_S$ was studied by Budd-Norbury [2], Budd [3], Merle-Peletier [8], Dolbeault-Flores [4], and Guo-Wei [6]. Note that (5) is transformed into (1) with $f(u) = u + u^p$ by changing $u \mapsto \lambda^{\frac{1}{p-1}} u$. According to [2, 4, 6], the bifurcation curve has infinitely many turning points if $p_S < p < p_{JL}$. In [6], for large solutions, the nonexistence of turning points was proved in a certain range of p ($> p_{JL}$).

First we show the uniqueness of the singular solution of the problem (1), and the behavior of the solutions $(\lambda(\alpha), u(r,\alpha))$ as $\alpha \to \infty$. By a singular solution u of (1), we mean that $u(r)$ is a classical solution of (1) for $0 < r \leq 1$ and satisfies $u(r) \to \infty$ as $r \to 0$.

Proposition 1. (i) *There exists a unique $\lambda^* > 0$ such that the problem* (1) *with $\lambda = \lambda^*$ has a singular solution u^*. The solution u^* is a unique singular solution of* (1) *with $\lambda = \lambda^*$. Furthermore, u^* satisfies*

$$u^*(r) = A(\sqrt{\lambda^*} r)^{-\theta}(1 + O(r^{\delta\theta})) \quad \text{as } r \to 0,$$

where $\delta > 0$ is the constant in (3),

$$\theta = \frac{2}{p-1} \quad \text{and} \quad A := \left\{ \frac{2}{p-1} \left(N - 2 - \frac{2}{p-1} \right) \right\}^{\frac{1}{p-1}}.$$

(ii) *Let $(\lambda(\alpha), u(r,\alpha))$ be a solution of* (1) *with $u(0,\alpha) = \alpha > 0$. Then, as $\alpha \to \infty$,*

$$\lambda(\alpha) \to \lambda^* \quad \text{and} \quad u(r,\alpha) \to u^*(r) \quad \text{in } C^1_{\mathrm{loc}}((0,1]),$$

where (λ^*, u^*) is the singular solution obtained in (i).

Following the idea by [9], we define three types of bifurcation diagrams according to the intersection number of $\lambda(\alpha)$ and λ^* for $\alpha > 0$. Let $I \subset \mathbf{R}$ be an interval, and let $f \in C(I)$. We define the zero-number of f in I by

$$\mathcal{Z}_I(f) = \sup\{n \in \mathbf{N} : \text{ there are } \alpha_1, \ldots, \alpha_{n+1} \in I,\ \alpha_1 < \cdots < \alpha_{n+1} \text{ such that } f(\alpha_i)f(\alpha_{i+1}) < 0 \text{ for } 1 \leq i \leq n\}$$

if f changes sign in I, and $\mathcal{Z}_I(f) = 0$ otherwise. By $\mathcal{T}[\mathcal{C}]$ we denote the number of the turning points of $\mathcal{C}$.

Definition. Put $m = \mathcal{Z}_{(0,\infty)}(\lambda(\cdot) - \lambda^*)$.

(i) We say that $\mathcal{C}$ is of Type I if $m = \infty$. As a consequence, if $\mathcal{C}$ is of Type I, then (1) has infinitely many regular solutions for $\lambda = \lambda^*$ and $\mathcal{T}[\mathcal{C}] = \infty$.

(ii) We say that $\mathcal{C}$ is of Type II if $m = 0$.

(iii) We say that $\mathcal{C}$ is of Type III if $1 \leq m < \infty$. As a consequence, $\mathcal{C}$ is of Type III if (1) has at least one and finitely many regular solutions for $\lambda = \lambda^*$.

For the classification of the bifurcation diagrams for (1), a partial result was obtained by [9] with the relation to the Morse index of the singular solution u^*. By $m(u^*)$ we define

$$m(u^*) = \sup\{\dim X : \ X \subset H^1_{0,\mathrm{rad}}(B),\ H[\phi] < 0 \text{ for all } \ \phi \in X \setminus \{0\}\},$$

where

$$H[\phi] = \int_B (|\nabla\phi|^2 - \lambda^* f'(u^*)\phi^2)dx.$$

It was shown by [9] that, if $p_S < p < p_{JL}$, then $m(u^*) = \infty$ and the curve $\mathcal{C}$ is of Type I, and that, if $N \geq 11$ and $p > p_{JL}$, then $0 \leq m(u^*) < \infty$.

In this talk, we will consider the classification of structure of solution curve of (1) in the case where $N \geq 11$ and $p \geq p_{JL}$ by means of the zero number of the solutions to

$$\phi'' + \frac{N-1}{r}\phi' + \lambda^* f'(u^*)\phi = 0 \quad \text{for } 0 < r < 1.$$

Furthermore, we will show some sufficient conditions on $f(u)$ for the bifurcation diagrams to be Type II and Type III.

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