GLOBAL DYNAMICS BELOW THE GROUND STATE FOR THE FOCUSING NONLINEAR SCHRÖDINGER EQUATION WITH A LINEAR REPULSIVE POTENTIAL

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1. Introduction

We study global dynamics of the solution to the Cauchy problem for the focusing semilinear Schrödinger equation with a linear potential on the real line $\mathbb{R}$:

$$(\mathrm{NLS}_V) \qquad \begin{cases} i\partial_t u + \partial_x^2 u - Vu + |u|^{p-1}u = 0, & (t,x) \in I \times \mathbb{R}, \\ u(0) = u_0 \in \mathcal{H}, \end{cases}$$

where $u = u(t,x)$ is a complex-valued unknown function of (t,x), $(0 \in)$ I denotes a existence time interval of the function u, $V \in L^1(\mathbb{R}) + L^\infty(\mathbb{R})$ is a non-negative repulsive potential, where L^p denotes a usual Lebesgue space, $p > 5$ belongs to the mass-supercritical region, $u_0 = u_0(x)$ is a complex-valued prescribed function of $x \in \mathbb{R}$, and $\mathcal{H}$ is a Hilbert space associated with the Schrödinger operator $-\partial_x^2 + V$ and is called energy space.

The Cauchy problem (NLS_V) is locally well-posed in the energy space $\mathcal{H}$.

Proposition 1.1 (Local well-posedness in $\mathcal{H}$)**.** *Let $V \in L^1(\mathbb{R}) + L^\infty(\mathbb{R})$ be non-negative and $p \geq 1$. Then the Cauchy problem (NLS_V) is locally well-posed in the energy space $\mathcal{H}$ for arbitrary initial data $u_0 \in \mathcal{H}$. By the existence result and the uniqueness result, the maximal existence times $T_\pm$ of the solution are well defined. Moreover, the conservation laws and the blow-up criterion hold:*

- *(Conservation Laws) The energy E and the mass M are conserved by the flow, i.e.*

$$E(u(t)) = E(u_0), \ M(u(t)) = M(u_0), \ \text{for any } t \in I_T,$$

where the functionals $E : \mathcal{H} \mapsto \mathbb{R}$ and $M : L^2(\mathbb{R}) \mapsto \mathbb{R}$ are defined as

$$E(\phi) = E_V(\phi) := \frac{1}{2}\|\phi\|^2_{H_V^{\frac{1}{2}}} - \frac{1}{p+1}\|\phi\|^{p+1}_{L^{p+1}}, \tag{1.1}$$

$$M(\phi) := \|\phi\|^2_{L^2}. \tag{1.2}$$

- *(Blow-up criterion) If $T_\pm < \infty$, then*

$$\lim_{t \to \pm T_\pm \mp 0} \|\partial_x u(t)\|_{L^2} = \infty,$$

where double-sign corresponds.

Our aim in the presentation is to study global dynamics of the solution and prove a scattering result and a blow-up result of the problem (NLS_V) with the initial data whose mass-energy is less than that of the ground state Q_1, where $Q_1 = Q_1(x)$ is the unique radial positive solution to the stationary Schrödinger equation without the potential:

$$-Q_1'' + Q_1 = |Q_1|^{p-1}Q_1, \ \text{in } H^1(\mathbb{R}). \tag{1.3}$$

Theorem 1.2 (Dichotomy between an upper bound and a lower bound of solutions)**.** *Let $p > 5$, $V \in L^1(\mathbb{R}) + L^\infty(\mathbb{R})$ be non-negative, $u_0 \in \mathcal{H}$ and u be the unique solution to (NLS_V) on the maximal interval $I = (-T_-, T_+)$. Moreover we assume that u_0 satisfies*

$$M(u_0)^\sigma E_V(u_0) < M(Q_1)^\sigma E_0(Q_1), \tag{1.4}$$

where $\sigma = \sigma(p) := \frac{p+3}{p-5}$ and $Q_1 \in H^1(\mathbb{R})$ is the unique positive radially symmetric solution to (1.3). Then the following holds:

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(1) *If u_0 satisfies*

$$\|u_0\|_{L^2}^{\sigma} \left\| H_V^{\frac{1}{2}} u_0 \right\|_{L^2} < \|Q_1\|_{L^2}^{\sigma} \|\partial_x Q_1\|_{L^2}, \tag{1.5}$$

then $T_+ = T_- = \infty$ and u satisfies

$$\|u_0\|_{L^2}^{\sigma} \left\| H_V^{\frac{1}{2}} u(t) \right\|_{L^2} < \|Q_1\|_{L^2}^{\sigma} \|\partial_x Q_1\|_{L^2},$$

for any $t \in \mathbb{R}$.

(2) *If u_0 satisfies*

$$\|u_0\|_{L^2}^{\sigma} \left\| H_V^{\frac{1}{2}} u_0 \right\|_{L^2} > \|Q_1\|_{L^2}^{\sigma} \|\partial_x Q_1\|_{L^2}, \tag{1.6}$$

then u satisfies

$$\|u_0\|_{L^2}^{\sigma} \left\| H_V^{\frac{1}{2}} u(t) \right\|_{L^2} > \|Q_1\|_{L^2}^{\sigma} \|\partial_x Q_1\|_{L^2},$$

for any $t \in I$.

Next we state a scattering result and a blow-up or grow-up result of solutions to (NLS$_V$) under some additional assumptions on the potential V, that is V is repulsive and integrable:

$$xV'(x) \leq 0, \text{ for a.e. } x \in \mathbb{R}, \text{ and } V \in L^1(\mathbb{R}), \text{ and } V' \in L^1_1(\mathbb{R}). \tag{A}$$

Theorem 1.3 (Scattering and blow-up or grow-up results below the ground state)**.** *In addition to the assumptions of Theorem 1.2, we assume that the potential V satisfies (A). Then the following holds:*

(1) *If u_0 satisfies (1.5) and the potential V belongs to $L^1_1(\mathbb{R})$, then the solution u scatters in $\mathcal{H}$ as $t \to \pm\infty$.*

(2) *If u_0 satisfies (1.6) and the potential V satisfies the inequality*

$$-xV'(x) - 2V(x) \leq 0 \quad \text{for a.e. } x \in \mathbb{R}, \tag{1.7}$$

then one of the following four cases occurs:

(a) *The solution u blows up in both time directions.*

(b) *The solution u blows up in a positive time, and u is global toward negative time and* $\limsup_{t\to-\infty} \|\partial_x u(t)\|_{L^2} = \infty$ *holds.*

(c) *The solution u blows up in a negative time, and u is global toward positive time and* $\limsup_{t\to\infty} \|\partial_x u(t)\|_{L^2} = \infty$ *holds.*

(d) *The solution u is global in both time directions and* $\limsup_{t\to\pm\infty} \|\partial_x u(t)\|_{L^2} = \infty$ *holds.*

The similar results for (NLS) with another potential were obtained in [3, 5], where [3] treats the focusing mass-supercritical NLS with a repulsive Dirac delta potential in one space dimension (see also [1] for the defocusing case) and [5] studied the focusing mass-supercritical NLS with a inverse square potential in three spatial dimensions or higher dimensions. On the other hand, Lafontaine [4] studied the defocusing version of (NLS$_V$), that is, (NLS$_V$) with a replacement of $+|u|^{p-1}u$ into $-|u|^{p-1}u$, and proved that the local solution solution can be extended globally and it tends to a free one as $t \to \pm\infty$ in the energy space $H^1(\mathbb{R})$ for an arbitrary data in $H^1(\mathbb{R})$. However, as for study of the classification of global behaviors of solutions, the focusing case are more difficult than the defocusing one, because the sign of the energy functional of the solution to the focusing problem changes by the size of the initial data.

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