Resolvent expansions for the Schrödinger operator on the discrete half-line

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This talk is based on a recent joint work [1] with Arne Jensen, Aalborg University. The purpose of this talk is to investigate relationship between the generalized eigenspaces and the expansion coefficients of resolvent for the Schrödinger operator $H = H_0 + V$ on the discrete half-line $\mathbb{N} = \{1, 2, \ldots\}$. Here H_0 is defined for any sequence $x \colon \mathbb{N} = \{1, 2, \ldots\} \to \mathbb{C}$ by

$$(H_0 x)[n] = \begin{cases} 2x[1] - x[2] & \text{for } n = 1, \\ 2x[n] - x[n+1] - x[n-1] & \text{for } n \geq 2, \end{cases}$$

and we assume that V is real-valued and that for some integer $\beta \geq 1$ and $\epsilon > 0$

$$V[n] = O(|n|^{-1-2\beta-\epsilon}).$$

Let us define the *generalized zero eigenspaces* by

$$\widetilde{\mathcal{E}} = \bigl\{\Psi \in \ell^{\infty,-\beta}(\mathbb{N});\ H\Psi = 0\bigr\}.$$

We can verify the specific asymptotics: $\widetilde{\mathcal{E}} \subset \mathbb{C}\mathbf{n} \oplus \mathbb{C}\mathbf{1} \oplus \ell^{1,\beta-2}(\mathbb{N})$, where $\mathbf{n}[m] = m$ and $\mathbf{1}[m] = 1$ for $m \in \mathbb{N}$, and hence the following subspaces make sense:

$$\mathcal{E} = \widetilde{\mathcal{E}} \cap \bigl(\mathbb{C}\mathbf{1} \oplus \ell^{1,\beta-2}(\mathbb{N})\bigr), \quad \mathsf{E} = \widetilde{\mathcal{E}} \cap \ell^{1,\beta-2}(\mathbb{N}).$$

Definition. The threshold $z = 0$ is said to be

1. a *regular point*, if $\mathcal{E} = \mathsf{E} = \{0\}$;
2. an *exceptional point of the first kind*, if $\mathcal{E} \supsetneq \mathsf{E} = \{0\}$;
3. an *exceptional point of the second kind*, if $\mathcal{E} = \mathsf{E} \supsetneq \{0\}$;
4. an *exceptional point of the third kind*, if $\mathcal{E} \supsetneq \mathsf{E} \supsetneq \{0\}$.

Now let us set $R(\kappa) = (H + \kappa^2)^{-1}$ and $\mathcal{B}^s = \mathcal{B}\bigl(\ell^{1,s}(\mathbb{N}), \ell^{\infty,-s}(\mathbb{N})\bigr)$.

Theorem 1. *Assume that the threshold* 0 *is a* regular *point, and that* $\beta \geq 2$. *Then*

$$R(\kappa) = \sum_{j=0}^{\beta-2} \kappa^j G_j + \mathcal{O}(\kappa^{\beta-1}) \quad \textit{in } \mathcal{B}^{\beta-2}$$

with $G_j \in \mathcal{B}^{j+1}$ *for* j *even, and* $G_j \in \mathcal{B}^j$ *for* j *odd.*

Theorem 2. *Assume that the threshold* 0 *is an exceptional point of the* first kind, *and that* $\beta \geq 3$. *Then*

$$R(\kappa) = \sum_{j=-1}^{\beta-4} \kappa^j G_j + \mathcal{O}(\kappa^{\beta-3}) \quad \textit{in } \mathcal{B}^{\beta-1}$$

with $G_j \in \mathcal{B}^{j+3}$ *for* j *even, and* $G_j \in \mathcal{B}^{j+2}$ *for* j *odd. In addition,*

$$G_{-1} = |\Psi_c\rangle\langle\Psi_c|,$$

where $\Psi_c \in \mathcal{E}$ *is the canonical resonance function.*

Theorem 3. *Assume that the threshold* 0 *is an exceptional point of the* second kind, *and that* $\beta \geq 4$. *Then*

$$R(\kappa) = \sum_{j=-2}^{\beta-6} \kappa^j G_j + \mathcal{O}(\kappa^{\beta-5}) \quad \textit{in } \mathcal{B}^{\beta-2}$$

with $G_j \in \mathcal{B}^{j+3}$ *for* j *even, and* $G_j \in \mathcal{B}^{j+2}$ *for* j *odd. In addition,*

$$G_{-2} = P_0, \quad G_{-1} = 0,$$

where P_0 *is the projection onto* E.

Theorem 4. *Assume that the threshold* 0 *is an exceptional point of the* third kind, *and that* $\beta \geq 4$. *Then*

$$R(\kappa) = \sum_{j=-2}^{\beta-6} \kappa^j G_j + \mathcal{O}(\kappa^{\beta-5}) \quad \textit{in } \mathcal{B}^{\beta-2}$$

with $G_j \in \mathcal{B}^{j+3}$ *for* j *even, and* $G_j \in \mathcal{B}^{j+2}$ *for* j *odd. In addition,*

$$G_{-2} = P_0, \quad G_{-1} = |\Psi_c\rangle\langle\Psi_c|,$$

where P_0 *is the projection onto* E, *and* $\Psi_c \in \mathcal{E}$ *is the canonical resonance function.*

References

[1] K. Ito and A. Jensen, *Resolvent expansions for the Schrödinger operator on the discrete half-line*, to appear in J. Math. Phys.