Quantum scattering in time-depeding electromagnetic fields

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Abstract

Consider the time-dependent magnetic Hamiltonian

$$H_0(t) = (p - qA(t,x))^2/(2m), \quad A(t,x) = (-B(t)x_2, B(t)x_1)/2, \tag{1}$$

where $x = (x_1, x_2) \in \mathbf{R}^2$, $p = (p_1, p_2) = -i(\partial_1, \partial_2)$, $m > 0$ and $q \in \mathbf{R}\backslash\{0\}$ are the position, the momentum, the mass and the charge of the particle, respectively. $B(t) \in L^\infty(\mathbf{R})$ is the intense of the magnetic field at the time t, and which satisfies the periodic condition

$$B(t+T) = B(t).$$

Denote a unitary propagator generated by $H_0(t)$ as $U_0(t,0)$.

Recently, the scattering theory for a time-periodic pulsed magnetic field was studied by Adachi-'K, (A.H.P. 2016), where the pulsed magnetic field is the followings

$$B(t) = \begin{cases} B, & t \in [NT, NT+T_B], \\ 0, & t \in [NT+T_B, (N+1)T], \end{cases}, \quad 0 < T_B < T, \ N \in \mathbf{Z}. \tag{2}$$

The following Lemma shows that the asymptotic behavior of the charged particle governed by the Hamiltonian (1) with respect to (2) is classified into the following three types accordingly to the B, T_B and T.

Lemma 1 (Adachi-K'). Let $\tilde{D} = 2\cos(qBT_B/(2m)) - qB(T-T_B)\sin(qBT_B/(2m))/2$ and $\tilde{\lambda}_\pm := \tilde{D}/2 \pm \sqrt{\tilde{D}^2/4 - 1}$. Then for all $\phi \in C_0^\infty(\mathbf{R}^2)$,

$$\begin{aligned} &\text{if } \tilde{D}^2 < 4, \ \|xU_0(NT,0)\phi\|_{(L^2(\mathbf{R}^2))^2} = \mathcal{O}(1), \\ &\text{if } \tilde{D}^2 = 4, \ \|xU_0(NT,0)\phi\|_{(L^2(\mathbf{R}^2))^2} = \mathcal{O}(|N|^1), \\ &\text{if } \tilde{D}^2 > 4, \ \|xU_0(NT,0)\phi\|_{(L^2(\mathbf{R}^2))^2} = \mathcal{O}(|\mu_N|), \quad \mu_N = (\tilde{\lambda}_+)^N - (\tilde{\lambda}_-)^N, \end{aligned} \tag{3}$$

hold as $N \to \pm\infty$.

Roughly speaking, the term $\|xU_0(t,0)\phi\|_{(L^2(\mathbf{R}^2))^2}$ can be regarded like the position of the particle at t, and hence (3) implies that the charged particle moves out the origin with the velocity growing exponentially in t, where we note that either $|\lambda_+|$ or $|\lambda_-|$ is larger than 1 if $\tilde{D}^2 > 4$. We call this phenomena *the particle is in exponentially scattering state* in this talk.

Moreover Korotyaev, (Math. USSR Sb., 1990) considered the same Hamiltonian $H_0(t)$ in (1) for time-periodic magnetic fields with conditions mentioned later and proved the asymptotic completeness of wave operators by using the following representation of the free propagator:

$$U_0(t,0) = e^{i(y_2'(t)/y_2(t))x^2} e^{-iy_1(t)y_2(t)p^2} e^{i\Omega(t)L} e^{-i\log(|y_2(t)|)A} \mathcal{S}^{\mathbf{n}(t)},$$

where $A_0 = (x \cdot p + p \cdot x)/2$ and $L = x_1p_2 - x_2p_1$ are called a generator of dilation group and the angular momentum of the charged particle, respectively, $y_j(t)$, $j \in \{1,2\}$ is the solution of

$$y_j''(t) + \left(\frac{qB(t)}{2m}\right)^2 y_j(t) = 0, \quad \begin{cases} y_1(0) = 0, \\ y_1'(0) = 1, \end{cases} \quad \begin{cases} y_2(0) = 1, \\ y_2'(0) = 0, \end{cases} \tag{4}$$

respectively, and $\mathcal{S}$ is an unitary operator which satisfies $(\mathcal{S}\phi)(x) = (-1)\phi(-x)$ and $\mathbf{n}(t)$ is the number of zeros of $\zeta_1(t)$ in $[0,t]$. Assumptions of magnetic fields in Korotyaev are the followings:
There exist $y_j(t)$ the solution of (4) such that

$$\begin{cases} y_1(t) & = ty_2(t) + \chi_1(t), \\ y_2(t) & = \chi_2(t), \end{cases} \quad \text{or} \quad \begin{cases} y_1(t) & = e^{\lambda t}\chi_1(t), \\ y_2(t) & = e^{-\lambda t}\chi_2(t), \end{cases} \tag{5}$$

hold, where $\lambda \in \mathbf{R}$ and χ_1 and χ_2 are periodic or antiperiodic functions.
The case where $y_j(t)$ is written by l.h.s. of (5) is closely related to the case of $\tilde{D} = 4$ in the pulsed magnetic field, and $y_j(t)$ is written by r.h.s. of (5) is closely related to (3) (i.e., exponentially scattering state).

Here we consider the *repulsive* Hamiltonian H_{rp}, that is,

$$H_{\text{rp}} := p^2/2 - \alpha^2 x^2/2,$$

where $\alpha \neq 0$. It is well known (see e.g. Bony-Carles-Häfner-Michel, J. Math.Pures Appl. 2006) that the classical trajectory satisfies

$$\left\| x e^{-itH_{\text{rp}}}\phi \right\| = \mathcal{O}(e^{|\alpha t|}),$$

and hence the particle goverened by H_{rp} is in exponentially scattering state.
Conjecture
There may exist some relations between $H_0(t)$ with r.h.s. of (5) (or $\tilde{D}^2 > 4$ in pulsed case) and H_{rp}.

Theorem('K, preprint)

Suppose that $B(t) \in L^\infty(\mathbf{R})$ and $B(t+T) = B(t)$ for some $T > 0$. Let $\zeta_1(t)$ and $\zeta_2(t)$ are the fundamental solutions of *Hill's* equation, i.e.,

$$\zeta_j''(t) + \left(\frac{qB(t)}{2m}\right)^2 \zeta_j(t) = 0, \quad \begin{cases} \zeta_1(0) & = 1, \\ \zeta_1'(0) & = 0, \end{cases} \quad \begin{cases} \zeta_2(0) & = 0, \\ \zeta_2'(0) & = 1, \end{cases}$$

and potential V be the multiplication operator with $(V\phi)(t,x) = V(x)\phi(t;x)$, $\phi \in L^2(\mathbf{R}/T\mathbf{Z}; L^2(\mathbf{R}^2))$, where $V(x) \in C^2(\mathbf{R}^2)$ satisfies

$$|\nabla^l V(x)| \leq C(1+|x|)^{-\rho_0}$$

for some $\rho_0 > 0$ and for all $l \in \{0,1,2\}$. Define the Floquet Hamiltonian acting on $L^2(\mathbf{R}/T\mathbf{Z}; L^2(\mathbf{R}^2))$ by

$$\hat{H} = -i\partial_t + H_0(t) + V,$$

and *discriminant of Hill's equation* D by

$$D = \zeta_1(T) + \zeta_2'(T).$$

Then there exists a unitary operator $\mathscr{I}_D(t)$ on $L^2(\mathbf{R}/T\mathbf{Z}; L^2(\mathbf{R}^2))$ such that

$$\mathscr{I}_D(t)^* \hat{H} \mathscr{I}_D(t) := \hat{W} = -i\partial_t + W_0 + \hat{V}(t;x,p), \quad \hat{V}(t;x,p) := \mathscr{I}_D(t)^* V \mathscr{I}_D(t)$$

holds, where *time-independent Hamiltonian* W_0 acting on $L^2(\mathbf{R}^2)$ can be written by

$$W_0 = (B_D/T)(p^2 - C_D^2 x^2 + D_D L) + (\pi\sigma_D)/T.$$

with some constants $B_D \neq 0$, $D_D \neq 0$ and $\sigma_D \in \{0,1\}$. In particular, C_D^2 satisfies

$$C_D^2 \equiv 0 \text{ if } D^2 = 4,$$
$$C_D^2 > 0 \text{ if } D^2 > 4.$$

Remark If $B(t)$ is pulsed, then D and $\tilde{D}$ in Lemma 1 are the same. Moreover, if $y_j(t)$ satisfies r.h.s. of (5) and $\zeta_j(t)$ satisfies $\zeta_1(t) = y_2(t)$ and $\zeta_2(t) = y_1(t)$, then at least, $D^2 > 4$.

Futhermore, in the case of $D^2 > 4$, we shall prove *Mourre estimate* for Hamiltonian $\hat{H}$.