A diffusive Lotka-Volterra prey-predator system with finitely many protection zones

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This talk is concerned with the following prey-predator system with protection zones for the prey:

$$
\text{(P)} \quad \begin{cases}
u_t = \Delta[(1 + k\rho(x)v)u] + u(\lambda - u - b(x)v) & \text{in } \Omega \times (0, \infty), \\
v_t = \Delta v + v(\mu - v + cu) & \text{in } \Omega \setminus \overline{\Omega}_0 \times (0, \infty), \\
\frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \times (0, \infty), \\
\frac{\partial v}{\partial n} = 0 & \text{on } \partial(\Omega \setminus \overline{\Omega}_0) \times (0, \infty), \\
u(x, 0) = u_0(x) \geq 0 & \text{in } \Omega, \\
v(x, 0) = v_0(x) \geq 0 & \text{in } \Omega \setminus \overline{\Omega}_0.
\end{cases}
$$

Here Ω is a bounded domain in $\mathbb{R}^N (N \geq 2)$ with smooth boundary $\partial\Omega$ and Ω_0 is an open subset of Ω with smooth boundary $\partial\Omega_0$; n is the outward unit normal vector on the boundary; $k \geq 0$, $\lambda > 0$, $\mu > 0$ and $c > 0$ are all constants; $\rho(x)$ is a smooth function in $\overline{\Omega}$ with $\partial\rho/\partial n = 0$ on $\partial\Omega$ and $b(x)$ is a Hölder continuous function in $\overline{\Omega}$. We make the following assumptions: $\rho(x) > 0$ and $b(x) > 0$ in $\overline{\Omega} \setminus \overline{\Omega}_0$; $\rho(x) = b(x) = 0$ in $\overline{\Omega}_0$; both $\rho(x)/b(x)$ and $b(x)/\rho(x)$ are bounded in $\overline{\Omega} \setminus \overline{\Omega}_0$;

$$
\Omega_0 = \bigcup_{i=1}^{\ell} O_i, \qquad \overline{O}_i \cap \overline{O}_j = \emptyset \text{ when } i \neq j, \tag{1}
$$

where each O_i is a simply connected open set satisfying $\overline{O}_i \subset \Omega$.

Unknown functions $u(x, t)$ and $v(x, t)$ denote the population densities of prey and predator respectively; λ and μ denote the intrinsic growth rates of the respective species. The diffusion term $\Delta[(1 + k\rho(x)v)u]$ means that the movement of the prey species in $\overline{\Omega} \setminus \overline{\Omega}_0$ is affected by population pressure from the predator species, and $k\Delta[\rho(x)vu]$ is referred to as a cross-diffusion term, which was originally proposed by Shigesada et al. [10].

For each i, the subregion O_i is called a protection zone because the predator species cannot enter Ω_0. Many researchers have studied the effect of a single protection zone (i.e. $\ell = 1$) on various population models (see [1, 3, 4, 5] for prey-predator models without cross-diffusion, [2] for a competition model without cross-diffusion, [6, 7, 8, 12] for prey-predator models with cross-diffusion, and [11] for a competition model with cross-diffusion).

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In this talk, we mainly discuss the stationary problem of (P) with $\ell \geq 1$. The stationary problem associated with (P) is

$$\text{(SP)} \quad \begin{cases} \Delta[(1 + k\rho(x)v)u] + u(\lambda - u - b(x)v) = 0 & \text{in } \Omega, \\ \Delta v + v(\mu - v + cu) = 0 & \text{in } \Omega \setminus \overline{\Omega}_0, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \\ \frac{\partial v}{\partial n} = 0 & \text{on } \partial(\Omega \setminus \overline{\Omega}_0). \end{cases}$$

We define

$$\lambda^*_\infty(k, \Omega_0) = \begin{cases} \displaystyle\inf_{\left\{\phi \in H^1(\Omega) \,:\, \int_{\Omega_0} \phi^2 dx > 0\right\}} \frac{\displaystyle\int_\Omega |\nabla\phi|^2 dx + \frac{1}{k}\int_{\Omega\setminus\overline{\Omega}_0} \frac{b(x)}{\rho(x)}\phi^2 dx}{\displaystyle\int_{\Omega_0} \phi^2 dx} & \text{if } k > 0, \\ \displaystyle\min_{i=1,2,\cdots,\ell} \lambda_1^D(O_i) & \text{if } k = 0, \end{cases} \tag{2}$$

where $\lambda_1^D(O_i)$ is the first eigenvalue of $-\Delta$ over O_i with the homogeneous Dirichlet boundary condition. Moreover, for $q \in L^\infty(\Omega)$, we denote by $\lambda_1^N(q, \Omega)$ the first eigenvalue of $-\Delta + q$ over Ω with the homogeneous Neumann boundary condition.

The stationary problem (SP) has three constant solutions $(u, v) = (0, 0)$, $(\lambda, 0)$ and $(0, \mu)$. Then we have the following theorem.

Theorem 1. *The following results hold true*:

(i) *Suppose that* $0 < \lambda < \lambda^*_\infty(k, \Omega_0)$. *Then there exists a positive number* μ^* *such that* $(0, \mu)$ *is unstable if* $0 < \mu < \mu^*$, *and asymptotically stable if* $\mu > \mu^*$. *Here* μ^* *is the unique positive solution of*

$$\lambda_1^N\left(\frac{b(x)\mu^* - \lambda}{1 + k\rho(x)\mu^*}, \Omega\right) = 0. \tag{3}$$

(ii) *Suppose that* $\lambda \geq \lambda^*_\infty(k, \Omega_0)$. *Then* $(0, \mu)$ *is unstable for any* $\mu > 0$.

(iii) *Both* $(0, 0)$ *and* $(\lambda, 0)$ *are unstable for any* $\lambda > 0$ *and any* $\mu > 0$.

We also have the following theorem.

Theorem 2. *The following results hold true*:

(i) *Suppose that* $0 < \lambda < \lambda^*_\infty(k, \Omega_0)$ *and let* μ^* *be the positive number defined by* (3). *Then* (SP) *has at least one positive solution if* $0 < \mu < \mu^*$, *and no positive solution if* $\mu \geq \mu^*$.

(ii) *Suppose that* $\lambda \geq \lambda^*_\infty(k, \Omega_0)$. *Then* (SP) *has at least one positive solution for any* $\mu > 0$.

Theorems 1 and 2 were obtained in [9]. Theorem 1 can be proved by analyzing the spectrum of the linearized operator around each constant solution. Theorem 2 can be proved by applying the bifurcation theory.

Theorems 1 and 2 assert that $\lambda^*_\infty(k, \Omega_0)$ is the critical prey growth rate for survival. We see from (1) and (2) that $\lambda^*_\infty(0, \Omega_0)$ does not necessarily decrease even if ℓ increases. This fact shows that not all of the protection zones are effectively utilized when $k = 0$.

References

[1] R. Cui, J. Shi and B. Wu, Strong Allee effect in a diffusive predator-prey system with a protection zone, J. Differential Equations **256** (2014), 108–129.

[2] Y. Du and X. Liang, A diffusive competition model with a protection zone, J. Differential Equations **244** (2008), 61–86.

[3] Y. Du, R. Peng and M. X. Wang, Effect of a protection zone in the diffusive Leslie predator-prey model, J. Differential Equations **246** (2009), 3932–3956.

[4] Y. Du and J. Shi, A diffusive predator-prey model with a protection zone, J. Differential Equations **229** (2006), 63–91.

[5] X. He and S. N. Zheng, Protection zone in a modified Lotka-Volterra model, Discrete Contin. Dyn. Syst. Ser. B **20** (2015), 2027–2038.

[6] K. Oeda, Effect of cross-diffusion on the stationary problem of a prey-predator model with a protection zone, J. Differential Equations **250** (2011), 3988–4009.

[7] K. Oeda, Coexistence states of a prey-predator model with cross-diffusion and a protection zone, Adv. Math. Sci. Appl. **22** (2012), 501–520.

[8] K. Oeda, Positive steady states for a prey-predator cross-diffusion system with a protection zone and Holling type II functional response, Discrete Contin. Dyn. Syst. Suppl. 2013 (2013), 597–603.

[9] K. Oeda, Steady-state solutions of a diffusive prey-predator model with finitely many protection zones, submitted.

[10] N. Shigesada, K. Kawasaki and E. Teramoto, Spatial segregation of interacting species, J. Theoret. Biol. **79** (1979), 83–99.

[11] Y.-X. Wang and W.-T. Li, Effect of cross-diffusion on the stationary problem of a diffusive competition model with a protection zone, Nonlinear Anal. Real World Appl. **14** (2013), 224–245.

[12] Y.-X. Wang and W.-T. Li, Effects of cross-diffusion and heterogeneous environment on positive steady states of a prey-predator system, Nonlinear Anal. Real World Appl. **14** (2013), 1235–1246.