Scattering theory and weak limit theorem in quantum walks [1]

1 Definition of the model

We consider a one-dimensional discrete time quantum walk, whose state evolution is described by

$$\Psi_t(x) = P(x+1)\Psi_{t-1}(x+1) + Q(x-1)\Psi_{t-1}(x-1) + R(x)\Psi_{t-1}(x), \quad (1)$$

where $(x,t) \in \mathbb{Z} \times \mathbb{N}$, Ψ_t is the state of a quantum walker at time t. The state Ψ_t is a normalized vector in the Hilbert space of states

$$\mathcal{H} = \ell^2(\mathbb{Z};\mathbb{C}^2) = \left\{ \Psi : \mathbb{Z} \to \mathbb{C}^2 \mid \sum_{x\in\mathbb{Z}} \|\Psi(x)\|^2_{\mathbb{C}^2} < \infty \right\}.$$

We write vectors $\Psi \in \mathcal{H}$ as $\Psi(x) = \begin{pmatrix} \Psi_1(x) \\ \Psi_2(x) \end{pmatrix} \in \mathbb{C}^2$ at each position $x \in \mathbb{Z}$. In order to write the state evolution (1) using a unitary evolution operator, we introduce a shift operator S and a coin operator C on $\mathcal{H}$ as follows. For all $\Psi \in \mathcal{H}$, the action of S is given by

$$(S\Psi)(x) = \begin{pmatrix} p\Psi_1(x) + q\Psi_2(x+1) \\ \bar{q}\Psi_1(x-1) - p\Psi_2(x) \end{pmatrix}, \quad x \in \mathbb{Z}, \quad (2)$$

where $p \in \mathbb{R}$ and $q \in \mathbb{C}$ satisfy $p^2 + |q|^2 = 1$. Let $\{C(x)\}_{x\in\mathbb{Z}} \subset U(2)$ be a family of 2×2 unitary matrices. The coin operator C is defined as a multiplication operator by $C(x)$. Let $U = SC$. If we take

$$P(x) = q\begin{pmatrix} 0 & 0 \\ a(x) & b(x) \end{pmatrix}, \quad Q(x) = \bar{q}\begin{pmatrix} c(x) & d(x) \\ 0 & 0 \end{pmatrix}, \quad R(x) = p\begin{pmatrix} a(x) & b(x) \\ -c(x) & -d(x) \end{pmatrix},$$

and

$$C(x) = \begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix},$$

then $\Psi_t = U^t\Psi_0$ ($t \in \mathbb{N}$) satisfy (1). Here, $\Psi_0 \in \mathcal{H}$ ($\|\psi_0\| = 1$) is the initial state of the walker.

[1] This work is in collaboration with T. Fuda and D. Funakawa.

2 Weak limit theorem

In quantum walks, the position X_t of the walker described by a random variable, which follows the probability distribution

$$P(X_t = x) = \|\Psi_t(x)\|^2_{\mathbb{C}^2}, \quad x \in \mathbb{Z}.$$

We suppose that there exist unitary matrices $C_\pm \in U(2)$ such that

$$C(x) = C_\pm + O(|x|^{-1-\epsilon}), \quad x \to \pm\infty$$

with some $\epsilon > 0$. Then, we have the following weak limit theorem.

Theorem 2.1 *Let X_t be as above. Then, X_t/t converges in law to a random variable V. Moreover, the following are equivalent:*

(1) The quantum walker is localized at some $x \in \mathbb{Z}$.

(2) $\lim_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T-1} P(X_t = x)$ at some $x \in \mathbb{Z}$.

(3) $P(V = 0) > 0$.

(4) U has an eigenvalue.

Here, we say that the quantum walker is localized at x if

$$\limsup_{t\to\infty} P(X_t = x) > 0.$$

3 Localization

By assumption, S is a unitary self adjoint operator. We assume that $C(x)$ are unitary self-adjoint and $\dim\ker(C(x) - 1) = 1$. Let

$$\chi(x) = \begin{pmatrix} \chi_1(x) \\ \chi_2(x) \end{pmatrix} \in \ker(C(x) - 1), \quad \|\chi(x)\|_{\mathbb{C}^2} = 1.$$

We suppose that $|p| \neq 1, \ \chi_1(x)\chi_2(x) \neq 0$. Then, we have a sufficient condition for the quantum walker to be localized.

Theorem 3.1 If

$$\limsup_{x\to+\infty} \left| \frac{(p \pm 1)\bar{\chi}_2(x)}{q\bar{\chi}_1(x)} \right| < 1, \quad \limsup_{x\to-\infty} \left| \frac{q\bar{\chi}_1(x)}{(p \pm 1)\bar{\chi}_2(x)} \right| < 1,$$

then the quantum walker is localized at all x.