Stability for line solitary waves of Zakharov–Kuznetsov equation

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We consider the Zakharov–Kuznetsov equation on $\mathbb{R} \times \mathbb{T}_L$:

$$\text{(ZK)} \qquad \begin{cases} u_t + \partial_x(\Delta u + u^2) = 0, & (t,x,y) \in \mathbb{R} \times \mathbb{R} \times \mathbb{T}_L, \\ u(0,x,y) = u_0(x,y), & (x,y) \in \mathbb{R} \times \mathbb{T}_L \end{cases}$$

where u is a real valued unknown function and $\mathbb{T}_L = \mathbb{R}/2\pi L\mathbb{Z}$. In [16], the Zakharov–Kuznetsov equation is derived the propagation of ionic-acoustic waves in uniformly magnetized plasma. The global well-posedness of Cauchy problem of (ZK) in the energy space $H^1(\mathbb{R} \times \mathbb{T}_L)$ was proved by Molinet–Pilod [10]. The equation (ZK) has solitary wave solutions which are defined by non-trivial solutions to (ZK) with the form

$$u(t,x,y) = Q(x - ct, y)$$

for some traveling speed $c > 0$. Then, $Q(x-ct, y)$ is a solitary wave if and only if Q is a non-trivial solution to the stationary equation

$$-\Delta Q + cQ - Q^2 = 0, \quad (x,y) \in \mathbb{R} \times \mathbb{T}_L. \tag{1}$$

We define the orbital stability of solitary wave of (ZK) as follows.

Definition 1. We say that a solitary wave $Q(x - ct, y)$ is orbitally stable if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for all initial data $u_0 \in H^1(\mathbb{R} \times \mathbb{T}_L)$ with $\|u_0 - Q\|_{H^1} < \delta$, the solution $u(t)$ to (ZK) with $u(0) = u_0$ satisfies

$$\sup_{t>0} \inf_{(x_0,y_0)\in\mathbb{R}\times\mathbb{T}_L} \|u(t,\cdot,\cdot) - Q(\cdot - x_0, \cdot - y_0)\|_{H^1} < \varepsilon.$$

Otherwise, we say the solitary wave $Q(x - ct, y)$ is orbitally unstable.

The orbital stability and the asymptotic stability of solitary waves of the Zakharov–Kuznetsov equation on $\mathbb{R}^2$ was proved in [4, 3].

If the solution u to (ZK) does not depend on the variable of the transverse direction $\mathbb{T}_L$, then u is also a solution to the Korteweg–de Vrise equation:

$$\text{(KdV)} \qquad u_t + u_{xxx} + 2uu_x = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}.$$

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The KdV equation has the soliton solution $R_c(t,x) = Q_c(x-ct)$, where

$$Q_c(x) = \frac{3c}{2}\cosh^{-2}\Big(\frac{\sqrt{c}x}{2}\Big).$$

The orbital stability of R_c for $c>0$ was showed in [1] and the asymptotic stability of R_c was proved in [12, 9, 7].

In this talk, we regard R_c as a line solitary wave of (ZK) and consider the orbital stability and the asymptotic stability of line solitary wave of (ZK). Namely, we consider the stability of line solitary wave

$$\tilde{R}_c(t,x,y) = R_c(t,x) = Q(x-ct), \quad (t,x,y) \in \mathbb{R}\times\mathbb{T}_L.$$

If $\tilde{R}_c$ is unstable, then we say that this instability is the transverse instability. For $c>0$, the linear instability of $\tilde{R}_c$ on $\mathbb{R}\times\mathbb{T}_L$ for large L was showed by Bridges [2]. On the whole space $\mathbb{R}^2$, The transverse instability of $\tilde{R}_c$ for $c>0$ was proved by Rousset–Tzvetkov [13]. On the periodical space $\mathbb{T}_{L_1}\times\mathbb{T}_{L_2}$, Johnson [6] proved the transverse instability of line periodic solitary waves.

Main result for the orbital stability is the following.

Theorem 2. *Let $L, c>0$. Then, the following hold.*

(i) *If $0 < L \le \frac{2}{\sqrt{5c}}$, then $\tilde{R}_c$ is orbitally stable on $\mathbb{R}\times\mathbb{T}_L$.*

(ii) *If $L > \frac{2}{\sqrt{5c}}$, then $\tilde{R}_c$ is orbitally unstable on $\mathbb{R}\times\mathbb{T}_L$.*

For $L>0$, $c=\frac{4}{5L^2}$ is the critical traveling speed which is the boundary between the orbital stability and the orbital instability. In the following proposition, we show $Q_{\frac{4}{5L^2}}$ is a bifurcation point on the branch of solutions Q_c.

Proposition 3. *Let $L>0$. Then, there exist $\delta_0>0$ and $\varphi_L \in C^2((-\delta_0,\delta_0)^2, H^2(\mathbb{R}\times\mathbb{T}_L))$ such that for $\boldsymbol{a}=(a_1,a_2)\in(-\delta_0,\delta_0)^2$ we have $\varphi_L(\boldsymbol{a})(x-\check{c}(\boldsymbol{a})t,y)$ is a stable solitary wave of (ZK) with*

$$\varphi_L(\boldsymbol{a})(x,y) = Q_{\frac{4}{5L^2}} + a_1 Q^{\frac{3}{2}}_{\frac{4}{5L^2}}\cos\frac{y}{L} + a_2 Q^{\frac{3}{2}}_{\frac{4}{5L^2}}\sin\frac{y}{L} + O(|\boldsymbol{a}|^2).$$

The following theorem is an main result for the asymptotic stability.

Theorem 4. *Let $c, L>0$. If $0<L<\frac{2}{\sqrt{5c}}$, then $\tilde{R}_c$ is asymptotically stable on $H^1(\mathbb{R}\times\mathbb{T}_L)$ in the sense by Martel–Merle [8]. i.e.(the critical case $c=\frac{4}{5L^2}$). For any $\beta>0$, there exists $\varepsilon_\beta>0$ such that for $u_0\in H^1(\mathbb{R}\times\mathbb{T}_L)$ with $\Big\|u_0 - Q_{\frac{4}{5L^2}}\Big\|_{H^1} < \varepsilon_\beta$, there exist $\rho_1(t),\rho_2(t)\in C^1([0,\infty),\mathbb{R})$, $c_+>0$ and $\boldsymbol{a}_+\in\mathbb{R}^2$ satisfying that*

$$\begin{cases} \big\|u(t,\cdot,\cdot) - Q_{c_+}(\cdot-\rho_1(t),\cdot-\rho_2(t))\big\|_{H^1(x>\beta t)} \to 0 \ \text{as}\ t\to\infty, \\ (\dot\rho_1(t),\dot\rho_2(t)) \to (c_+,0) \ \text{as}\ t\to\infty \end{cases}$$

or

$$\begin{cases} \|u(t,\cdot,\cdot) - \varphi_L(\boldsymbol{a}_+)(\cdot-\rho_1(t),\cdot-\rho_2(t))\|_{H^1(x>\beta t)} \to 0 \ \text{as}\ t\to\infty, \\ (\dot\rho_1(t),\dot\rho_2(t)) \to (\check{c}(\boldsymbol{a}_+),0) \ \text{as}\ t\to\infty, \end{cases}$$

where

$$\|u\|_{H^1(x>\beta t)}^2 = \int_{\{x>\beta t\}} (|\nabla u|^2 + |u|^2) dxdy.$$

Moreover,

$$\left|\frac{4}{5L^2} - c_+\right| + |\boldsymbol{a}_+|^2 \lesssim \left\|u_0 - Q_{\frac{4}{5L^2}}\right\|_{H^1}.$$

In general, Q_c is not a ground state of the stationary equation (1). Therefore, to prove the orbital stability of $\tilde{R}_c$ on $\mathbb{R} \times \mathbb{T}_L$, we can not apply the variational method. For $0 < L < \frac{2}{\sqrt{5c}}$, we show the orbital stability of $\tilde{R}_c$ by applying the Lyapunov function method by [5]. In general, the number of the negative eigenvalue of the linearized operator for the stationary equation (1) around Q_c is larger than one. Thus, we can not show the orbital instability of $\tilde{R}_c$ by applying the Lyapunov function method by [5]. For $L > \frac{2}{\sqrt{5c}}$, we show the orbital instability of $\tilde{R}_c$ on $\mathbb{R} \times \mathbb{T}_L$ from the linear instability of the linearized equation of (ZK) around $\tilde{R}_c$ by applying the argument in [14]. To prove the linear instability, we use the method of Evans' function in [11]. In the critical case $L = \frac{2}{\sqrt{5c}}$, the linearized operator of (1) around Q_c is degenerate. To prove the orbital stability in the critical case, we apply the argument in [15].

For the proof of the asymptotic stability of $\tilde{R}_c$, we apply the argument by Martel–Merle [7, 8] for KdV equation and Côte–Muñoz–Pilod–Simpson [3]. This argument relies on a Liouville type theorem for spatially decaying solutions around a solitary wave. From the orbital stability and the monotonicity property, solutions near by a solitary wave converge to an exponentially decaying function in $H^1(x > a)$. From the Liouville type theorem, this function must be solitary wave. The virial type estimate is important to show the Liouville type theorem. In the case $0 < L < \frac{2}{\sqrt{5c}}$, the linearized operator of the stationary equation (1) around Q_c is coercive on $\{u \in H^1 : \|u\|_{L^2} = \|Q_c\|_{L^2}\}$ by modulating translation. Using the coerciveness, we can show the Liouville type theorem from the usual virial type estimate. However, in the case $L = \frac{2}{\sqrt{5c}}$, this linearized operator is degenerate. Therefore, we can not show the Liouville type theorem by applying the usual virial type estimate. To show the Liouville type theorem for $L = \frac{2}{\sqrt{5c}}$, we use modulated solitary waves and the modulated virial type estimate. In this talk, I show the outline of proof of Theorem 4 and explain the argument to prove the asymptotic stability.

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