Solvability and singular optimal control problems for quasi-variational evolution equations

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§1 Introduction

This is a joint work with Nobuyuki KENMOCHI (Chiba University, Chiba, Japan) and Ken SHIRAKAWA (Chiba University, Chiba, Japan).

In this talk, we consider the following double quasi-variational evolution equations governed by time-dependent subdifferentials in the Banach space V^*:

$$\text{(QP)} \begin{cases} \partial_*\psi^t(u;u'(t)) + \partial_*\varphi^t(u;u(t)) + g(t,u(t)) \ni f(t) \text{ in } V^* \text{ for a.a. } t \in (0,T), \\ u(0) = u_0 \text{ in } V, \end{cases}$$

where V is a uniformly convex Banach space such that V is dense in a real Hilbert space H and the injection from V into H is compact. We also suppose that the dual space V^* of V is uniformly convex, and $H = H^*$. In addition, $0 < T < \infty$, $u' = du/dt$ in V, $g(t,\cdot)$ is a single-valued Lipschitz operator in V^*, f is a given V^*-valued function, and $u_0 \in V$ is a given initial data. The time-dependent function $\psi^t(v;z)$ is proper, lower semi-continuous (l.s.c.), and convex in $z \in V$. Also, $\varphi^t(v;z)$ is a time-dependent, non-negative, continuous convex function in $z \in V$. Note that $(t,v) \in [0,T] \times C([0,T];H)$ is a parameter that determines the convex functions $\psi^t(v;\cdot)$ and $\varphi^t(v;\cdot)$ on V. In addition, the subdifferentials $\partial_*\psi^t(v;z)$ of $\psi^t(v;z)$ with respect to $z \in V$ is a multivalued operator in V^*, and $\partial_*\varphi^t(v;z)$ of $\varphi^t(v;z)$ with respect to $z \in V$ is a single-valued linear operator in V^*.

The main aim of this talk is to establish the solvability result for (QP). We also investigate a singular optimal control problem formulated for non-well-posed state systems (QP).

§2 Main Theorem

We suppose that the duality mapping $F : V \to V^*$ is strongly monotone. In addition, we assume the following conditions *(A)*, *(B)* and *(C)*.

(*Assumption A*)

Let ψ_0^t be a proper, l.s.c., and convex function on V for each $t \in [0,T]$, and setting

$$D_0 := \left\{ v \in W^{1,2}(0,T;V) \;\middle|\; \int_0^T \psi_0^t(v'(t))dt < \infty \right\},$$

we define a functional $\psi^t : [0,T] \times D_0 \times V \to \mathbb{R}$ such that $\psi^t(v;z)$ is proper, l.s.c., and convex in $z \in V$ for any $t \in [0,T]$ and any $v \in D_0$, and

$$\psi^t(v_1;z) = \psi^t(v_2;z), \ \forall z \in V, \text{ if } v_1 = v_2 \text{ on } [0,t],$$

for $v_i \in D_0, \ i = 1,2$. We assume the following:

(A1) If $\{t_n\}_{n\in\mathbb{N}} \subset [0,T]$, $t \in [0,T]$, $\{v_n\}_{n\in\mathbb{N}} \subset D_0$, $\sup_{n\in\mathbb{N}} \int_0^T \psi_0^t(v_n'(t))dt < \infty$, $v \in D_0$, $t_n \to t$ and $v_n \to v$ in $C([0,T];H)$ as $n \to \infty$, then $\psi^{t_n}(v_n;\cdot) \to \psi^t(v;\cdot)$ on V in the sense of Mosco [2] as $n \to \infty$.

(A2) $D(\psi^t(v;\cdot))$ is a non-empty subset of $D(\psi_0^t)$ for all $v \in D_0$ and $t \in [0,T]$, and

$$\psi^t(v;z) \geq \psi_0^t(z), \quad \forall t \in [0,T], \ \forall v \in D_0, \ \forall z \in D(\psi^t(v;\cdot)).$$

Moreover, there exist positive constants $C_1 > 0$ and $C_2 > 0$ such that

$$\psi_0^t(z) \geq C_1 |z|_V^2 - C_2, \quad \forall t \in [0,T],\ \forall z \in D(\psi_0^t).$$

(A3) $\partial_* \psi^t(v;0) \ni 0$ for all $t \in [0,T]$ and $v \in D_0$, and there is a non-negative function $c_\psi(\cdot) \in L^1(0,T)$ such that $\psi^t(v;0) \leq c_\psi(t)$ for a.a. $t \in [0,T],\ \forall v \in D_0$.

(*Assumption B*)

We define a functional $\varphi^t : [0,T] \times D_0 \times V \to \mathbb{R}$ such that $\varphi^t(v;z)$ is non-negative, finite, continuous, and convex in $z \in V$ for any $t \in [0,T]$ and any $v \in D_0$, and

$$\varphi^t(v_1;z) = \varphi^t(v_2;z),\ \forall z \in V,\ \text{if } v_1 = v_2 \text{ on } [0,t],$$

for $v_i \in D_0,\ i = 1,2$. We assume the following:

(B1) The subdifferential $\partial_* \varphi^t(v;z)$ of $\varphi^t(v;z)$ with respect to $z \in V$ is linear and bounded from $D(\partial_* \varphi^t(v;\cdot)) = V$ into V^* for each $t \in [0,T]$ and $v \in D_0$, and there is a positive constant C_3 such that

$$|\partial_* \varphi^t(v;z)|_{V^*} \leq C_3 |z|_V, \quad \forall z \in V,\ \forall t \in [0,T],\ \forall v \in D_0.$$

(B2) If $\{v_n\}_{n\in\mathbb{N}} \subset D_0$, $\sup_{n\in\mathbb{N}} \int_0^T \psi_0^t(v_n'(t))dt < \infty$, $v \in D_0$ and $v_n \to v$ in $C([0,T];H)$ (as $n \to \infty$), then $\partial_* \varphi^t(v_n;z) \to \partial_* \varphi^t(v;z)$ in V^*, $\forall z \in V,\ \forall t \in [0,T]$ as $n \to \infty$.

(B3) $\varphi^0(v;0) = 0$ for all $v \in D_0$. Moreover, there is a positive constant C_4 such that

$$\varphi^0(v;z) \geq C_4 |z|_V^2, \quad \forall z \in V,\ \forall v \in D_0.$$

(B4) There is a function $\alpha \in W^{1,1}(0,T)$ such that

$$|\varphi^t(v;z) - \varphi^s(v;z)| \leq |\alpha(t) - \alpha(s)| \varphi^s(v;z), \quad \forall z \in V,\ \forall v \in D_0,\ \forall s,t \in [0,T].$$

(*Assumption C*)

Let g be a single-valued operator from $[0,T] \times V$ into V^* such that $g(t,z)$ is strongly measurable in $t \in [0,T]$ for each $z \in V$, and assume:

(C1) For each $t \in [0,T]$, the operator $z \to g(t,z)$ is continuous from V_w into V^*, i.e., if $z_n \to z$ weakly in V as $n \to \infty$, then $g(t,z_n) \to g(t,z)$ in V^* as $n \to \infty$.

(C2) $g(t,\cdot)$ is uniformly Lipschitz from V into V^*, i.e., there is a constant $L_g > 0$ such that

$$|g(t,z_1) - g(t,z_2)|_{V^*} \leq L_g |z_1 - z_2|_V, \quad \forall t \in [0,T],\ \forall z_i \in V\ (i = 1,2).$$

Main Theorem. Suppose that Assumptions *(A), (B),* and *(C)* are satisfied. Let f be any function in $L^2(0,T;V^*)$, and u_0 be any element in V. Then, (QP;f, u_0) admits at least one solution u on $[0,T]$.

References

[1] N. Kenmochi, K. Shirakawa and N. Yamazaki, Double quasi-variational evolution equations governed by time-dependent subdifferentials and its singular optimal control, preprint.

[2] U. Mosco, Convergence of convex sets and of solutions variational inequalities, *Advances Math.,* **3** (1969), 510–585.