A sufficient condition for boundedness in a chemotaxis system with signal-dependent sensitivities modeling tumor angiogenesis*

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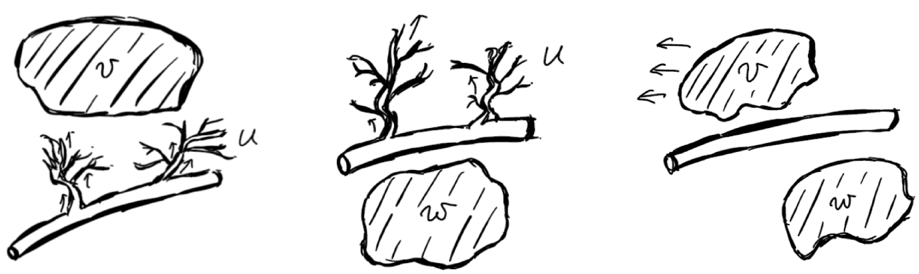
1. Introduction

One of the subjects of studies in mathematical biology is to analyze behavior of solutions to chemotaxis systems. Here *chemotaxis* is the property such that a species reacts on some chemical substance and moves towards or away from this substance. Chemotaxis systems describing such phenomena were proposed by Keller–Segel [**3**]. After that, many types of chemotaxis systems have been studied (e.g. Tello–Winkler [**5**], Winkler [**6, 7**]). In particular, the chemotaxis system for tumor angiogenesis,

$$\begin{cases} u_t = \Delta u - \nabla\cdot(u\nabla v) + \chi\nabla\cdot(u\nabla w), \\ v_t = \Delta v + \nabla\cdot(v\nabla w) - v + u, \\ 0 = \Delta w - w + u, \end{cases}$$

where $\chi > 0$ is a constant, was investigated. In this system the functions u, v, w represent the density of endothelial cells, the concentration of an enzyme, the density of an extracellular matrix, respectively. Also, the term $-\nabla\cdot(u\nabla v)$ means that blood vessels grow towards an enzyme (see Figure 1 (a)), the term $+\nabla\cdot(u\nabla w)$ idealizes that blood vessels move away from an extracellular matrix (see Figure 1 (b)), the term $+\nabla\cdot(v\nabla w)$ represents that an enzyme moves away from an extracellular matrix (see Figure 1 (c)).



(a) **(b)** **(c)**

Figure 1: The meanings of terms appearing in the above system

From a mathematical point of view, Tao–Winkler [**4**] gave a pioneering study of the parabolic–parabolic–*elliptic* system and established boundedness under largeness condition for χ. However, there is still scope for study on a fully parabolic i.e. parabolic–parabolic–*parabolic* system including sensitivity functions. The purpose of this talk is to derive boundedness in a fully parabolic system including sensitivity functions.

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In this talk we consider the fully parabolic chemotaxis system with signal-dependent sensitivities,

$$
\begin{cases}
u_t = \Delta u - \nabla \cdot (u\chi_1(v)\nabla v) + \nabla \cdot (u\chi_2(w)\nabla w), & x \in \Omega,\ t > 0,\\
v_t = \Delta v + \nabla \cdot (v\xi(w)\nabla w) + \alpha u - \beta v, & x \in \Omega,\ t > 0,\\
w_t = \Delta w + \gamma u - \delta w, & x \in \Omega,\ t > 0,\\
\nabla u \cdot \nu = \nabla v \cdot \nu = \nabla w \cdot \nu = 0, & x \in \partial\Omega,\ t > 0,\\
u(x,0) = u_0(x),\ v(x,0) = v_0(x),\ w(x,0) = w_0(x), & x \in \Omega,
\end{cases}
\tag{1.1}
$$

where $\Omega \subset \mathbb{R}^N$ ($N \le 3$) is a bounded domain with smooth boundary $\partial\Omega$; ν is the outward normal vector to $\partial\Omega$; u_0, v_0, w_0 are initial data fulfilling

$$
u_0 \in C^0(\overline{\Omega}) \setminus \{0\}, \quad v_0, w_0 \in W^{1,\infty}(\Omega) \quad \text{and} \quad u_0, v_0, w_0 \ge 0 \text{ in } \Omega. \tag{1.2}
$$

Also, χ_1, χ_2, ξ are positive known functions satisfying

$$
\chi_i, \xi \in C^{1+\vartheta}([0,\infty))\ (0 < \exists\,\vartheta < 1), \quad \chi_i, \xi > 0 \quad (i \in \{1,2\}), \tag{1.3}
$$

$$
\sup_{s>0} s\chi_i(s) < \infty \quad (i \in \{1,2\}), \tag{1.4}
$$

$$
\exists\, K_i > 0; \quad \chi_i'(s) + K_i|\chi_i(s)|^2 \le 0 \quad \text{for all } s \ge 0 \quad (i \in \{1,2\}) \tag{1.5}
$$

$$
\exists\, \xi_0 > 0; \quad \xi(s) \le \xi_0 \chi_2(s) \quad \text{for all } s \ge 0, \tag{1.6}
$$

$$
\exists\, K_3 > 0; \quad \xi'(s) \le K_3 \quad \text{for all } s \ge 0. \tag{1.7}
$$

The example of χ_1, χ_2, ξ are as follows:

$$
\chi_1(s) = \frac{a_1}{(b_1+s)^{k_1}}, \quad \chi_2(s) = \frac{a_2}{(b_2+s)^{k_2}}, \quad \xi(s) = \frac{a_3}{(b_3+s)^{k_3}} \quad \text{for } s \ge 0
$$

with $a_i > 0$, $b_i \ge 0$, $k_i > 1$. In this example the constants K_i in (1.5) are given by $\frac{b_i k_i}{a_i}$. Therefore some largeness conditions for K_1, K_2 imply that a_1, a_2 are small.

The main result reads as follows.

Theorem 1.1. *Assume that χ_1, χ_2 satisfy (1.3)–(1.5) with K_1, K_2 fulfilling*

$$
K_1 > \min_{\lambda \in J} \frac{2(2\lambda+1)(3\lambda+4) + \sqrt{D_\lambda}}{2(K_2-4) - \lambda^2 - 2}, \quad K_2 > 4 + \sqrt{2},
$$

where $J := \left(K_2 - 4 - \sqrt{(K_2-4)^2 - 2},\ K_2 - 4 + \sqrt{(K_2-4)^2-2}\right)$ and where

$$
D_\lambda := 4\lambda\big[2\,(K_2-4) + 3\lambda + 8\big]\big[\lambda\,(K_2-4) + 4\lambda^2 + 12\lambda + 7\big].
$$

Then there exists a constant $\xi_0^* > 0$ such that for all ξ satisfying (1.3), (1.6) with some $\xi_0 \in (0, \xi_0^*)$ as well as (1.7), and all (u_0, v_0, w_0) fulfilling (1.2) there exists a unique triplet (u, v, w) of nonnegative functions

$$
\begin{aligned}
&u \in C^0(\overline{\Omega} \times [0,\infty)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)),\\
&v, w \in \bigcap_{q>n} C^0([0,\infty); W^{1,q}(\Omega)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)),
\end{aligned}
$$

which solves (1.1) in the classical sense. Also, the solution is bounded in the sense that

$$
\|u(\cdot,t)\|_{L^\infty(\Omega)} \le C
$$

for all $t > 0$ with some $C > 0$.

2. Strategy for the proof

The strategy in the proof of Theorem 1.1 is to obtain L^2-boundedness of u and L^r-boundedness of ∇v with some $r > n$. The former can be shown by constructing the differential inequality

$$\frac{d}{dt}\int_\Omega u^2(x,t)f(x,t)\,dx \le c_1\int_\Omega u^2(x,t)f(x,t)\,dx - c_2\Big(\int_\Omega u^2(x,t)f(x,t)\,dx\Big)^{1+\theta}$$

for some constants $c_1, c_2, \theta > 0$ and some test function f; this method is based on a testing argument which was recently developed in the papers [**1, 2**]. Once we obtain L^2-boundedness of u, the next step is to prove L^r-boundedness of ∇v with some $r > n$. An L^r-estimate for ∇v can be shown by semigroup estimates in the case that $\xi = 0$; however, in the case that $\xi \neq 0$ this method breaks down due to the term $\nabla \cdot (v\xi(w)\nabla w)$. On the other hand, the method for the case that ξ is a constant in [**4**] does not work, because new complicated terms appear. In order to overcome this difficulty we first observe L^2-boundedness of ∇v by an energy estimate. We next upgrade L^2-boundedness of ∇v to L^r-boundedness of ∇v with some $r > n$. Finally, in light of L^r-boundedness of ∇v, we can obtain L^∞-boundedness of u, which yields global existence and boundedness.

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