Abstract evolution equations governed by time-dependent dissipative operators with respect to metric-like functionals

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1. Main theorem

Let X be a real Banach space and we consider the initial value problem

$$u'(t) = A(t)u(t) \quad \text{for } t \in [s, b), \qquad \text{and} \qquad u(s) = x,$$

where $-\infty < a \leq s < b \leq \infty$, $x \in X$ and $A = \{A(t); t \in [a, b)\}$ is a family of operators in X satisfying the dissipativity condition and the subtangential condition with growth condition described below.

Let $\varphi = \{\varphi^t; t \in [a, b)\}$ be a family of functionals on X to $[0, \infty]$ satisfying the following condition:

(φ) If $(t, x) \in [a, b)\times X$ and if $\{(t_n, x_n)\}$ is a sequence in $[a, b)\times X$ such that $x_n \in D(\varphi^{t_n})$ for $n \geq 1$, $\limsup_{n\to\infty} \varphi^{t_n}(x_n) < \infty$, $t_n \leq t$ for $n \geq 1$ and $(t_n, x_n) \to (t, x)$ as $n \to \infty$, then $x \in D(\varphi^t)$ and $\varphi^t(x) \leq \limsup_{n\to\infty} \varphi^{t_n}(x_n)$.

Let Φ be a nonnegative functional on $[a, b) \times X \times X$ satisfying the following conditions:

(Φ1) There exists a positive continuous function L on $[a, b)$ such that

$$|\Phi(t, x, y) - \Phi(t, \hat{x}, \hat{y})| \leq L(t)(\|x - \hat{x}\| + \|y - \hat{y}\|) \qquad \text{for } (x, y), (\hat{x}, \hat{y}) \in X \times X \text{ and } t \in [a, b).$$

(Φ2) $\Phi(t, x, x) = 0$ for $t \in [a, b)$ and $x \in D(\varphi^t)$.

(Φ3) If $t \in [a, b)$ and $(x, y) \in D(\varphi^t)\times D(\varphi^t)$ and if $\{(t_n, x_n, x_n)\}$ is a sequence in $[a, b)\times X\times X$ such that $t_n \leq t$ for $n \geq 1$, $(x_n, y_n) \in D(\varphi^{t_n}) \times D(\varphi^{t_n})$ for $n \geq 1$, $\limsup_{n\to\infty} \varphi^{t_n}(x_n) < \infty$, $\limsup_{n\to\infty} \varphi^{t_n}(y_n) < \infty$, and $(t_n, x_n, y_n) \to (t, x, y)$ in $[a, b) \times X \times X$ as $n \to \infty$, then

$$\Phi(t, x, y) \leq \limsup_{n\to\infty} \Phi(t_n, x_n, y_n).$$

(Φ4) If $\{(x_n, y_n)\}$ is a sequence in $X \times X$ and if there exist $r \geq 0$ and a sequence $\{t_n\}$ in $[a, b)$ such that $(x_n, y_n) \in D_r(\varphi^{t_n}) \times D_r(\varphi^{t_n})$ for $n \geq 1$, and $\{t_n\}$ converges in $[a, b)$ and $\Phi(t_n, x_n, y_n) \to 0$ as $n \to \infty$, then $\|x_n - y_n\| \to 0$ as $n \to \infty$.

Let $g \in C([a, b)\times[0, \infty); \boldsymbol{R})$ and $g(t, 0) \geq 0$ for $t \in [a, b)$. For each $(s, p) \in [a, b)\times[0, \infty)$, let $\tau(s, p)$ be the maximal existence time of the noncontinuable maximal solution $m(t; s, p)$ of the problem

$$p'(t) = g(t, p(t)) \quad \text{for } s \leq t < b, \qquad \text{and} \qquad p(s) = p.$$

We make the following assumptions on the family A.

(A1) $D(A(t)) = D(\varphi^t)$ for $t \in [a, b)$.

(A2) If $t \in [a, b)$ and $x \in D(\varphi^t)$ and if $\{(t_n, x_n)\}$ is a sequence in $[a, b) \times X$ such that $x_n \in D(\varphi^{t_n})$ for $n \geq 1$, $(t_n, x_n) \to (t, x)$ in $[a, b) \times X$ as $n \to \infty$ and $\limsup_{n\to\infty} \varphi^{t_n}(x_n) \leq \varphi^t(x)$, then

$$\lim_{n\to\infty} A(t_n)x_n = A(t)x \quad \text{in } X.$$

(A3) For each $r \geq 0$ there exists a nonnegative continuous function ω_r on $[a, b)$ such that

$$\liminf_{h\to0+} h^{-1}(\Phi(t + h, x + hA(t)x, y + hA(t)y) - \Phi(t, x, y)) \leq \omega_r(t)\Phi(t, x, y) \qquad \text{for } t \in [a, b) \text{ and } x, y \in D_r(\varphi^t).$$

(A4) For each $\epsilon > 0$, $t \in [a, b)$ and $x \in D(\varphi^t)$ there exist $h \in (0, \epsilon]$ and $x_h \in D(\varphi^{t+h})$ such that $t + h < b$,

$$\|x + hA(t)x - x_h\| \leq h\epsilon \quad \text{and} \quad h^{-1}(\varphi^{t+h}(x_h) - \varphi^t(x)) \leq g(t, \varphi^t(x)) + \epsilon.$$

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The following is the main theorem, which gives a generalization of [3, Theorem I].

Theorem 1. *Assume that for any $p \geq 0$ there exists $M > 0$ such that $t \in [a, b)$, $x \in D(\varphi^t)$ and $\varphi^t(x) \leq p$ imply $\|A(t)x\| \leq M$. Let $s \in [a, b)$ and $x \in D(\varphi^s)$, and set $\tau_0 = \tau(s, \varphi^s(x)) - s$. Then the initial value problem*

$$u'(t) = A(t)u(t) \quad \text{for } t \in [s, s + \tau_0), \quad \text{and} \quad u(s) = x$$

has a unique solution u in the sense that $u(t)$ is continuous for $t \in [s, s + \tau_0)$, $u(s) = x$, $u(t)$ is right-differentiable for $t \in [s, s + \tau_0)$, the right-derivative $(d/dt)^+u(t)$ is right-continuous for $t \in [s, s + \tau_0)$, $(d/dt)^+u(t) = A(t)u(t)$ for $t \in [s, s + \tau_0)$ and $\varphi^t(u(t)) \leq m(t; s, \varphi^s(x))$ for $t \in [s, s + \tau_0)$.

Remark 2. (i) The family φ is used to define the local quasi-dissipativity of the family A and specify the growth of a solution u in terms of the real-valued function $\varphi^t(u(t))$. In case of concrete partial differential equations the use of such a family φ corresponds to a priori estimates or energy estimates which ensure the global existence of the solutions as well as their asymptotic properties. The idea of the localization with respect to φ is affected by the Lyapunov method and our setting is similar in spirit to that of Oharu and Takahashi [5] discussing nonlinear semigroups associated with semilinear evolution equations. Another role of φ is to control the behavior of the domain of $A(t)$ with respect to t. (ii) A quasi-dissipativity condition in terms of a metric-like functional was found by Okamura [6] as a criterion for the uniqueness of solutions of ordinary differential equations. Condition (A3) is a slightly modified version of Kobayashi *et.* [2] where the mapping $(t, x) \to A(t)x$ is assumed to be continuous in X. Condition (Φ3) is about continuity of Φ in some sense. Condition (Φ4) roughly means that the topology induced by the functional Φ is equivalent to that induced by the original norm on each revel set of φ^t. The advantage of this condition is that it allows us to handle the case where the continuous dependence of solutions on their initial data is Hölder continuous. A subtangential condition was found by Nagumo [4] to characterize viability, and we use a subtangential condition combined with growth condition (see (A4)). (iii) The following consideration leads us to condition (A2): Let X be a Banach space such that both X and its dual space are uniformly convex, and let $\{\mathfrak{A}(t); t \in [a, b)\}$ be a family of maximal dissipative operators in X satisfying a type of demiclosed condition stated as follows: If $t \in [a, b)$ and $y \in X$ and if $t_n \in [a, b)$, $x_n \in D(\mathfrak{A}(t_n))$ and $y_n \in \mathfrak{A}(t_n)x_n$ for $n \geq 1$, and $t_n \to t$ and $y_n \to y$ weakly in X as $n \to \infty$, then $x \in D(\mathfrak{A}(t))$ and $y \in \mathfrak{A}(t)x$. This condition was used by Martin [3] with a "directional growth" condition, which is a special version of condition (A4). Let $A = \{A(t); t \in [a, b)\}$ be a family of the minimal sections of $\{\mathfrak{A}(t); t \in [a, b)\}$ and let $\varphi = \{\varphi^t; t \in [a, b)\}$ be a family of proper functionals from X into $[0, \infty]$ defined by $\varphi^t(x) = \|A(t)x\|$ if $x \in D(A(t))$, and $\varphi^t(x) = \infty$ otherwise. Then condition (φ) and condition (A2) can be proved to be satisfied. The dissipativity in the usual sense is the special case of (Φ4) with $\Phi(t, x, y) = \|x - y\|$ for $(t, x, y) \in [a, b) \times X \times X$ and $\omega_r = 0$.

We illustrate how Theorem 1 apply to the mixed problem for the inhomogeneous equation of Kirchhoff type

$$\begin{cases} u_{tt} = \beta'(\|\nabla u\|^2)\Delta u - \kappa u_t + f & \text{in } \Omega \times (0, T), \\ u_t + \beta'(\|\nabla u\|^2)\dfrac{\partial u}{\partial n} + \sigma u = g & \text{on } \Gamma \times (0, T) \end{cases}$$

for small initial data and forcing terms. From Theorem 1 we also derive the well-posedness for an abstract time-dependent linear evolution equation governed by operators whose domains do not necessarily contain a dense subspace of X differently in Kato [1], and the result applies to a degenerate evolution equation of the form

$$u''(t) + A(\phi(t)^2 u(t)) = 0 \quad \text{for } t \in [0, T)$$

in a real Hilbert space H, where A is a selfadjoint operator in H such that $\langle Au, u \rangle \geq 0$ for $u \in D(A)$, and ϕ satisfies the following condition: There exist $C > 0$ and $\beta > 0$ such that $C^{-1}t^\beta \leq \phi(t) \leq Ct^\beta$ for $t \in [0, T)$. An interesting result was proved by Shigeta [7] that the sum of the regularity of u and u' is kept for any $t \in [0, T)$, although the degeneracy occurs at $t = 0$. We can show continuity and right-differentiability at $t = 0$.

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