Maximal regularity for the Stokes equations with various boundary conditions

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1 Introduction

We consider the following resolvent and non-stationary Stokes equations with Dirichlet or Neumann boundary conditions in the half space $\mathbb{R}^n_+ = \mathbb{R}^{n-1} \times (0,\infty)$.

$$\begin{cases} \lambda u - \Delta u + \nabla \pi = f & \text{in } \mathbb{R}^n_+, \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}^n_+, \end{cases} \qquad \begin{cases} \partial_t U - \Delta U + \nabla \Pi = F & \text{in } \mathbb{R}^n_+ \times (0,\infty), \\ \operatorname{div} U = 0 & \text{in } \mathbb{R}^n_+ \times (0,\infty), \end{cases}$$

$$\text{(D)} \begin{cases} u_j = h_j \quad (j = 1, \ldots, n-1), \\ u_n = 0, \end{cases} \qquad \begin{cases} U_j = H_j \quad (j = 1, \ldots, n-1), \\ U_n = 0, \end{cases}$$

$$\text{(N)} \begin{cases} -(\partial_n u_j + \partial_j u_n) = h_j \quad (j = 1, \ldots, n-1), \\ -(2\partial_n u_n - \pi) = h_n \end{cases} \qquad \begin{cases} -(\partial_n U_j + \partial_j U_n) = H_j \quad (j = 1, \ldots, n-1), \\ -(2\partial_n U_n - \Pi) = H_n. \end{cases}$$

Notation : Let $\varepsilon \in (0, \pi/2)$, $1 < p, q < \infty$, $\gamma_0 \geq 0$, $s \geq 0$ and

$$\begin{aligned} \Sigma_\varepsilon &:= \{\lambda \in \mathbb{C} \setminus \{0\} \mid |\arg \lambda| < \pi - \varepsilon\}, \\ \hat{W}^1_q(D) &:= \{\pi \in L_{q,\mathrm{loc}}(D) \mid \nabla \pi \in L_q(D)\}, \\ L_{p,0,\gamma_0}(\mathbb{R}, X) &:= \{f : \mathbb{R} \to X \mid e^{-\gamma_0 t} f(t) \in L_p(\mathbb{R}, X),\ f(t) = 0 \text{ for } t < 0\}, \\ W^m_{p,0,\gamma_0}(\mathbb{R}, X) &:= \{f \in L_{p,0,\gamma_0}(\mathbb{R}, X) \mid e^{-\gamma_0 t} \partial_t^j f(t) \in L_p(\mathbb{R}, X),\ j = 1, \ldots, m\}, \\ H^s_{p,0,\gamma_0}(\mathbb{R}, X) &:= \{f : \mathbb{R} \to X \mid \Lambda^s_\gamma f := \mathcal{L}^{-1}_\lambda[|\lambda|^s \mathcal{L}[f](\lambda)](t) \in L_{p,0,\gamma}(\mathbb{R}, X) \text{ for any } \gamma \geq \gamma_0\}, \end{aligned}$$

where $\mathcal{L}$ and $\mathcal{L}^{-1}_\lambda$ are two-sided Laplace transform; let $\lambda = \gamma + i\tau \in \mathbb{C}$ and

$$\mathcal{L}[f](\lambda) = \int_{-\infty}^{\infty} e^{-\lambda t} f(t)\,dt = \mathcal{F}_{t\to\tau}(e^{-\gamma t} f), \quad \mathcal{L}^{-1}_\lambda[g](t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\lambda t} g(\lambda)\,d\tau = e^{\gamma t} \mathcal{F}_{\tau\to t} g.$$

2 Main theorems

Theorem 2.1 (resolvent L_q estimate). *Let $0 < \varepsilon < \pi/2$ and $1 < q < \infty$. Then for any $\lambda \in \Sigma_\varepsilon$,*

$$f \in L_q(\mathbb{R}^n_+), \quad h \in \begin{cases} W^2_q(\mathbb{R}^n_+) \text{ Dirichlet}, \\ W^1_q(\mathbb{R}^n_+) \text{ Neumann}, \end{cases}$$

there exists a unique solution $(u, \pi) \in W^2_q(\mathbb{R}^n_+) \times \hat{W}^1_q(\mathbb{R}^n_+)$ of resolvent Stokes equations with the estimate

$$\|(\lambda u, \lambda^{1/2} \nabla u, \nabla^2 u, \nabla \pi)\|_{L_q(\mathbb{R}^n_+)} \leq \begin{cases} C\|(f, \lambda h, \lambda^{1/2} \nabla h, \nabla^2 h)\|_{L_q(\mathbb{R}^n_+)} \text{ (D)}, \\ C\|(f, \lambda^{1/2} h, \nabla h)\|_{L_q(\mathbb{R}^n_+)} \text{ (N)}. \end{cases}$$

Theorem 2.2 (Maximal L_p-L_q estimate). *Let $1 < p, q < \infty$ and $\gamma_0 \geq 0$. Then for any*

$$F \in L_{p,0,\gamma_0}(\mathbb{R}, L_q(\mathbb{R}^n_+)), \quad H \in \begin{cases} W^1_{p,0,\gamma_0}(\mathbb{R}, L_q(\mathbb{R}^n_+)) \cap L_{p,0,\gamma_0}(\mathbb{R}, W^2_q(\mathbb{R}^n_+)) \text{ Dirichlet}, \\ H^{1/2}_{p,0,\gamma_0}(\mathbb{R}, L_q(\mathbb{R}^n_+)) \cap L_{p,0,\gamma_0}(\mathbb{R}, W^1_q(\mathbb{R}^n_+)) \text{ Neumann}, \end{cases}$$

and $U_0 = 0$, there exists a unique solution

$$U \in W^1_{p,0,\gamma_0}(\mathbb{R}, L_q(\mathbb{R}^n_+)) \cap L_{p,0,\gamma_0}(\mathbb{R}, W^2_q(\mathbb{R}^n_+)),$$
$$\Pi \in L_{p,0,\gamma_0}(\mathbb{R}, \hat{W}^1_q(\mathbb{R}^n_+))$$

of non-stationary Stokes equations with the estimate; for any $\gamma \geq \gamma_0$,

$$\|e^{-\gamma t}(U_t, \gamma U, \Lambda_\gamma^{1/2}\nabla U, \nabla^2 U, \nabla\Pi)\|_{L_p(\mathbb{R},L_q(\mathbb{R}^n_+))} \leq \begin{cases} C\|e^{-\gamma t}(F, H_t, \Lambda_\gamma^{1/2}\nabla H, \nabla^2 H)\|_{L_p(\mathbb{R},L_q(\mathbb{R}^n_+))} \text{ (D)}, \\ C\|e^{-\gamma t}(F, \Lambda_\gamma^{1/2} H, \nabla H)\|_{L_p(\mathbb{R},L_q(\mathbb{R}^n_+))} \text{ (N)}. \end{cases}$$

As the corollary of theorem 2.1, we have that the two kinds of Stokes operator generate bounded analytic semi-groups on solenoidal vector spaces. In theorem 2.2, we can take general initial data U_0 from above semi-group theory.

3 Strategy of the proofs

At first, we remark that the theorems have already proved in [1](Dirichlet) and [3, 4](Neumann). However, the main point is that we have shortened their proofs by using H^∞-calculus methods (c.f.[2]). (Here H^∞ means holomorphic and bounded.) In particular we proved the following new Fourier multiplier theorem for a suitable form which is used to the half space; Let us define the operators T and $\tilde{T}_\gamma$ by

$$T[m]f(x) = \int_0^\infty [\mathcal{F}^{-1}_{\xi'} m(\xi', x_n + y_n)\mathcal{F}_{x'} f](x, y_n)dy_n,$$
$$\tilde{T}_\gamma[m_\lambda]g(x,t) = \mathcal{L}^{-1}_\lambda \int_0^\infty [\mathcal{F}^{-1}_{\xi'} m_\lambda(\xi', x_n + y_n)\mathcal{F}_{x'}\mathcal{L}g](x, y_n)dy_n,$$
$$= [e^{\gamma t}\mathcal{F}^{-1}_{\tau\to t} T[m_\lambda]\mathcal{F}_{t\to\tau}(e^{-\gamma t}g)](x,t).$$

Let $\tilde{\Sigma}_\eta := \{z \in \mathbb{C} \setminus \{0\} \mid |\arg z| < \eta\} \cup \{z \in \mathbb{C} \setminus \{0\} \mid \pi - \eta < |\arg z|\}$ for $\eta \in (0, \pi/2)$.

Theorem 3.1. (i) *Let m satisfy the following two conditions:*
(a) *There exists $\eta \in (0, \pi/2)$ such that $\{m(\cdot, x_n), x_n > 0\} \subset H^\infty(\tilde{\Sigma}^{n-1}_\eta)$.*
(b) *There exist $\eta \in (0, \pi/2)$ and $C > 0$ such that $\sup_{\xi' \in \tilde{\Sigma}^{n-1}_\eta} |m(\xi', x_n)| \leq C x_n^{-1}$ for all $x_n > 0$.*
Then $T[m]$ is a bounded linear operator on $L_q(\mathbb{R}^n_+)$ for every $1 < q < \infty$.
(ii) *Let $\gamma_0 \geq 0$ and let m_λ satisfy the following two conditions:*
(c) *There exists $\eta \in (0, \pi/2 - \varepsilon)$ such that for each $x_n > 0$ and $\gamma \geq \gamma_0$,*

$$\tilde{\Sigma}^n_\eta \ni (\tau, \xi') \mapsto m_\lambda(\xi', x_n) \in \mathbb{C}$$

is bounded and holomorphic.
(d) *There exist $\eta \in (0, \pi/2 - \varepsilon)$ and $C > 0$ such that $\sup\{|m_\lambda(\xi', x_n)| \mid (\tau, \xi') \in \tilde{\Sigma}^n_\eta\} \leq C x_n^{-1}$ for all $\gamma \geq \gamma_0$ and $x_n > 0$.*
Then $\mathcal{F}^{-1}_{\tau\to t} T[m_\lambda]\mathcal{F}_{t\to\tau}$ is a bounded linear operator on $L_p(\mathbb{R}, L_q(\mathbb{R}^n_+))$, i.e., $\tilde{T}_\gamma[m_\lambda]$ satisfies

$$\|e^{-\gamma t}\tilde{T}_\gamma g\|_{L_p(\mathbb{R},L_q(\mathbb{R}^n_+))} \leq C\|e^{-\gamma t} g\|_{L_p(\mathbb{R},L_q(\mathbb{R}^n_+))}$$

for every $\gamma \geq \gamma_0$ and $1 < p, q < \infty$.

By this theorem, it is enough to check the order of the symbols although the previous papers need to decompose the symbols into some types.

We can show the resolvent L_q estimate as follows.

Step1. We show that $\operatorname{div} u = g = 0$ and $f = 0$ are enough by considering the problems in $\mathbb{R}^n$.
Step2. We find the multiplier symbols by considering partial Fourier transform $(h(x', 0) \to u)$.
Step3. By fundamental theorem, we make an integral of x_n. (Integrands are h and $\partial_n h$.)
Step4. We decompose the symbols so that we can use the multiplier theorem.
The proof of maximal L_p-L_q is similar to the resolvent estimate through Laplace transform.

Remark 3.2. *By a similar method, we are able to consider various boundary conditions* (*Robin, two-phase problem, on layer domain, with surface tension*).

References

[1] T. Kubo and Y. Shibata, Nonlinear differential equations, Asakura Shoten, Tokyo, 2012, (in Japanese).

[2] J. Prüss and G. Simonett, Moving Interfaces and Quasilinear Parabolic Evolution Equations, Birkhauser Monographs in Mathematics, 2016, ISBN: 978-3-319-27698-4

[3] Y. Shibata and S. Shimizu, On a resolvent estimate for the Stokes system with Neumann boundary condition, Diff. Int. Eqns. **16** (4) (2003), 385–426.

[4] Y. Shibata and S. Shimizu, On the maximal L_p-L_q regularity of the Stokes problem with first order boundary condition; model problems, J. Math. Soc. Japan **64** (2) (2012), 561–626.