Global boundedness in a degenerate tumor invasion system with chemotaxis effect

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We consider the following initial-boundary value problem of chemotaxis model:

$$
\text{(1)} \qquad \begin{cases} u_t = \nabla \cdot (D(u)\nabla u - S(u)\nabla v), & x \in \Omega,\ t > 0, \\ v_t = \Delta v + wz, & x \in \Omega,\ t > 0, \\ w_t = -wz, & x \in \Omega,\ t > 0, \\ z_t = \Delta z - z + u, & x \in \Omega,\ t > 0, \\ (D(u)\nabla u - S(u)\nabla v) \cdot \nu = \nabla v \cdot \nu = \nabla z \cdot \nu = 0, & x \in \partial\Omega,\ t > 0, \\ (u, v, w, z)(x, 0) = (u_0, v_0, w_0, z_0)(x), & x \in \Omega, \end{cases}
$$

where $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary $\partial\Omega$. Here $D \in C^{1+\theta}([0,\infty))$ (with some $\theta \in (0,1)$), $S \in C([0,\infty))$ are supposed to be nonnegative function, and the initial data satisfies

$$
\text{(2)} \qquad (u_0, v_0, w_0, z_0) \in L^\infty(\Omega) \times W^{1,\infty}(\Omega) \times W^{1,\infty}(\Omega) \times W^{1,\infty}(\Omega),\ u_0, v_0, w_0, z_0 \geq 0.
$$

The model (1) was proposed as a modified tumor invasion model with chemotaxis effect ([2]).

Known results. As know results, global-in-time boundedness ($\exists C$ s.t. $\forall t > 0; \|u(\cdot,t)\|_{L^\infty(\Omega)} \leq C$) was proved in each of the following cases:

- **[2, 3]:** $D(u) = 1$, $S(u) = u$, $N \leq 3$,
- **[1, 5]:** $D(u) \geq u^{m-1}$ $(m > 1)$, $S(u) = u$, $N \leq 3$,
- **[7]:** $D(u) = 1$, $S(u) = u$, $N = 4$ under small initial data,
- **[8]:** $D(u) \geq (u+1)^{m-1}$ $(m \in \mathbb{R})$, $0 \leq S(u) \leq (u+1)^\alpha$ $(\alpha \in \mathbb{R})$ with the condttion

$$
\alpha + 1 \leq \begin{cases} m + 1 + \frac{1}{N} & (1 \leq N \leq 3), \\ m + \frac{4}{N} & (N \geq 4). \end{cases}
$$

The proof in previous studies utilizes the uniform boundedness of $\|\nabla v(\cdot,t)\|_{L^p(\Omega)}$ $(1 < p < \infty)$, which is obtained from the L^1-conservation law of u and L^p-L^q estimates in the case of $N \leq 3$. On the other hand, in the case of $N \geq 4$, the combined energy of $u, \nabla v, \nabla z$ is used. There is a gap in the conditions for m, α due to the difference in these methods (see [8]). Also, as noted in [8, Remark 1.2], nothing is known when $\alpha + 1 \geq m + \frac{4}{N}$. Therefore, in the present study, we will consider the existence and boundedness of global weak solutions under the condition that $\alpha + 1 = m + \frac{4}{N}$ holds.

Definition. A quadruple (u, v, w, z) of non-negative functions defined on $\Omega \times [0,\infty)$ is called a global weak solution of (1) if for any $T > 0$,

- $u \in L^\infty(0,T; L^\infty(\Omega))$, $\int_0^u D(\sigma)\,d\sigma \in L^2(0,T; H^1(\Omega))$, $S(u) \in L^2(0,T; L^2(\Omega))$,
- $v \in L^\infty(0,T; W^{1,\infty}(\Omega))$,

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† S. Ishida is supported by JSPS Grant-in-Aid for Scientists Research (No. 21K13815).

- $w, z \in L^\infty(0,T; L^\infty(\Omega)) \cap L^2(0,T; H^1(\Omega))$,
- (u, v, w, z) fulfills (1) in the distributional sense.

Theorem ([6])

Supppose that (u_0, v_0, w_0, z_0) satisfy (2) and D, S satisfy

$$\exists m > 1 \text{ s.t. } \forall \sigma \geq 0;\ f(0) = 0,\ f(\sigma) \geq \sigma_1^{m-1},$$
$$\exists \alpha \in \mathbb{R} \text{ s.t. } \forall \sigma \geq 0;\ \begin{cases} g(\sigma) \leq \sigma^\alpha \text{ (if } \alpha \geq 0), \\ g(\sigma) \leq \sigma(\sigma+1)^{\alpha-1} \text{ (if } \alpha < 0). \end{cases}$$

1. If
$$\alpha + 1 = m + \frac{4}{N}\ (N \geq 3),$$
then there exists a constant $\delta = \delta(m, \alpha, N) > 0$ such that if u_0 has the additional property $\|u_0\|_{L^1(\Omega)} \leq \delta$ there exists a global weak solution (u, v, w, z) of (1) satisfying
$$\text{(3)} \qquad \begin{cases} u \in C_{w-*}([0,\infty); L^\infty(\Omega)), \\ \|u(t)\|_{L^\infty(\Omega)} \leq u_{\max}\ (\forall t \geq 0), \\ \|u(t)\|_{L^1(\Omega)} = \|u_0\|_{L^1(\Omega)}\ (\forall t \geq 0), \\ u(t) \to \overline{u_0} \quad \text{weakly* in } L^\infty(\Omega)\ (t \to \infty), \end{cases}$$
where $u_{\max} \geq 0$ is a constant which does not depend on t, and $\overline{u_0} := \frac{1}{|\Omega|}\int_\Omega u_0(x)\,dx$.
2. If
$$\alpha + 1 < m + \frac{4}{N}\ (N \geq 2),$$
then there exists a global weak solution (u, v, w, z) of (1) which fullfils (3).

In the one-dimensional case, we showed to global boundedness under $\alpha + 1 < m + 3$ ([4]).

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