FRACTIONAL SCHRÖDINGER EQUATION WITH THE DERIVATIVE CUBIC NONLINEARITY

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We consider the Cauchy problem for the critical derivative fractional nonlinear Schrödinger equation

$$
\left\{\begin{array}{c} i\partial_t u - \frac{1}{\alpha}\left|\partial_x\right|^{\alpha} u = i\partial_x\left(\left|u\right|^2 u\right),\ t>0, x\in\mathbb{R}, \\ u(0,x) = u_0(x),\ x\in\mathbb{R}, \end{array}\right. \tag{1}
$$

where $\alpha \in (1,2)$ and $\left|\partial_x\right|^{\alpha} = (-\partial_x^2)^{\alpha/2}$. Equation (1) with $\alpha = 2$ is the derivative nonlinear Schrödinger equation which was studied extensively, see, e.g., [1], [2]. The exceptional point $\alpha = 1$ corresponds to the so-called derivative half-wave equation.

Notation and Function Spaces. $\mathbf{L}^p = \{\phi \in \mathbf{S}'; \|\phi\|_{\mathbf{L}^p} < \infty\}$ is the usual Lebesgue space with norm $\|\phi\|_{\mathbf{L}^p} = \left(\int_{\mathbb{R}} |\phi(x)|^p dx\right)^{\frac{1}{p}}$ for $1 \le p < \infty$ and $\|\phi\|_{\mathbf{L}^\infty} = \text{ess.sup}_{x\in\mathbb{R}} |\phi(x)|$ for $p = \infty$. The weighted Sobolev space is

$$
\mathbf{H}_p^{m,s} = \left\{\varphi \in \mathbf{S}'; \|\phi\|_{\mathbf{H}_p^{m,s}} = \|\langle x\rangle^s \langle i\partial_x\rangle^m \phi\|_{\mathbf{L}^p} < \infty\right\},
$$

$m, s \in \mathbb{R}$, $1 \le p \le \infty$, $\langle x\rangle = \sqrt{1+x^2}$, $\langle i\partial_x\rangle = \sqrt{1-\partial_x^2}$. We also use the notations $\mathbf{H}^{m,s} = \mathbf{H}_2^{m,s}$, $\mathbf{H}^m = \mathbf{H}^{m,0}$ for simplicity. Let $\mathbf{C}(\mathbf{I};\mathbf{Y})$ be the space of continuous functions from the time interval $\mathbf{I}$ to a Banach space $\mathbf{Y}$.

The main result of this talk is the following.

Theorem 1. *Let $\alpha \in (1,2)$. Suppose that the initial data $u_0 \in \mathbf{H}^{3,1} \cap \mathbf{H}^4$ and $x\partial_x u_0 \in \mathbf{H}^{1,1} \cap \mathbf{H}^2$ are such that $\|u_0\|_{\mathbf{H}^{3,1}\cap\mathbf{H}^4} + \|x\partial_x u_0\|_{\mathbf{H}^{1,1}\cap\mathbf{H}^2} \le \varepsilon$. Then there exists an $\varepsilon > 0$ such that the Cauchy problem (1) has a unique global in time solution $u \in \mathbf{C}\left([0,\infty);\mathbf{H}^2\right)$ satisfying $\|u\|_{\mathbf{X}_\infty} \le C\varepsilon$, where*

$$
\begin{aligned}
\|u\|_{\mathbf{X}_\infty} &= \sup_{t\in[0,\infty)} \left\|\langle\xi\rangle^{\frac{3}{2}} \widehat{\varphi}\right\|_{\mathbf{L}^\infty} \\
&+ \sup_{t\in[0,\infty)} \langle t\rangle^{-\gamma} \left(\left\|\langle\xi\rangle^2 \widehat{\varphi}\right\|_{\mathbf{L}^2} + \left\|\langle\xi\rangle \widehat{\mathcal{P}}\widehat{\varphi}\right\|_{\mathbf{L}^2} + \right. \\
&\left. \left\|\langle\xi\rangle^2 \partial_\xi \widehat{\varphi}\right\|_{\mathbf{L}^2} + \left\|\langle\xi\rangle \partial_\xi \widehat{\mathcal{P}}\widehat{\varphi}\right\|_{\mathbf{L}^2}\right),
\end{aligned}
$$

$\widehat{\varphi} = \mathcal{F}\mathcal{U}(-t)\,u(t)$, $\widehat{\mathcal{P}} = \alpha t\partial_t - \xi\partial_\xi$ and $\gamma > 0$ is small depending on the size of the data. Moreover for any u_0 such that $\|u_0\|_{\mathbf{H}^{3,1}\cap\mathbf{H}^4} + \|x\partial_x u_0\|_{\mathbf{H}^{1,1}\cap\mathbf{H}^2} \le \varepsilon$, there

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exists a unique modified final state $W_+ \in \mathbf{L}^\infty$ *and* $\delta > 0$ *such that the asymptotics*

$$\begin{aligned} u(t,x) &= \frac{1}{\sqrt{t}} e^{-it\left(\frac{1}{\alpha}-1\right)\left|\frac{x}{t}\right|^{\frac{\alpha}{\alpha-1}}} W_+\left(t^{-\frac{1}{\alpha-1}} \frac{x}{|x|^{\frac{\alpha-2}{\alpha-1}}}\right) \\ &\times \exp\left(it^{-\frac{1}{\alpha-1}} \frac{x}{|x|^{\frac{\alpha-2}{\alpha-1}}} \left|W_+\left(\frac{x}{|x|^{\frac{\alpha-2}{\alpha-1}}}\right)\right|^2 \log t\right) \\ &+ O\left(\varepsilon t^{-\frac{1}{2}-\delta}\right) \end{aligned}$$

is valid for $t \to \infty$ *uniformly with respect to* $x \in \mathbb{R}$.

References

[1] Z. Guo, N.Hayashi, Y. Lin and P. Naumkin, *Modified scattering operator for the derivative nonlinear Schrödinger equation*, SIAM J. Math. Aanl., **45** (2013), No. 6, pp. 3854-3871.

[2] N.Hayashi and T.Ozawa, *Modified wave operators for the derivative nonlinear Schödinger equations*, Math. Annalen, **298** (1994), pp. 557-576.

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